Coalitional equilibria in non-cooperative games with strategic substitutes: Self-enforcing coalition deviations and irreversibility*

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Abstract

Introducing a *coalitional equilibrium with restricted deviations*, we examine how effectively equilibria based on coalitional stability refine Nash equilibria in games with σ-strategic substitutes and σ-monotone externalities. From the existing equilibria such as coalition-proof Nash equilibria and near-strong Nash equilibria, we can consider several ways to restrict coalitional deviations. We incorporate two natural self-enforcing conditions of coalition deviations, *Nash stability* and *irreversibility*, into the coalitional equilibrium and provide a more general analysis than earlier studies. We find it impossible that in each of the two stability concepts, the coalitional equilibrium effectively refines the Nash equilibrium for all games with σ-strategic substitutes and σ-monotone externalities.

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1 Introduction

We study the refinement of Nash equilibria in a strategic-form game with strategic substitutes (SS) and monotone externalities (ME). Since this game has many examples of economic games such as the Cournot oligopoly game and the game of the private provision of public goods, it is important from the viewpoint of applied game theory to clarify characteristics of the equilibria of this game. The Nash equilibrium, the standard equilibrium concept of the strategic-form game, is not necessarily uniquely determined in this game.[1] Hence, we apply "coalitional refinements" of the Nash equilibrium to the game.

Yi (1999) is the first study to apply the *coalition-proof Nash equilibrium* (Bernheim et al., 1987) to a class of games with SS and ME. Yi (1999) shows that every Pareto-undominated pure-strategy Nash equilibrium is coalition-proof. Shinohara (2010) shows that in the same game, the set of coalition-proof Nash equilibria coincides with the entire set of pure-strategy Nash equilibria. Quartieri and Shinohara (2015) clarify many properties of the coalition-proof Nash equilibria in σ-interactive games with σ-strategic substitutes (σ-SS) and σ-monotone externalities (σ-ME), which generalize Yi's (1999) and Shinohara's (2010) games. Quartieri and Shinohara (2015) show that the set of coalition-proof Nash equilibria under strong Pareto dominance (sCPN equilibria, for short) and the entire set of Nash equilibria coincide in these games. They also examine coalition-proof Nash equilibria under weak Pareto dominance (wCPN equilibria, for short) and show that the set of wCPN equilibria also coincides with the set of Nash equilibria if the best reply correspondence of all players is at most singleton-valued in the same games. Another familiar equilibrium to refine the Nash equilibria is a *strong Nash equilibrium* (Aumann, 1959). However, since the strong Nash equilibrium is too demanding, the set of strong Nash equilibria may be empty, although the set of Nash equilibria is nonempty in the games of Quartieri and Shinohara (2015). Therefore, it seems difficult that the familiar equilibria based on coalitional stability single out a particular Nash equilibrium from multiple Nash equilibria for games with σ-SS and σ-ME.

In this study, we examine whether the equilibrium based on coalitional stability that is both weaker than the strong Nash equilibrium and stronger than the coalition-proof Nash equilibrium effectively refines the Nash equilibrium. Some "intermediate" equilibrium concepts already exist. We can take a *semi-strong Nash equilibrium* (Kaplan, 1992; Milgrom and Roberts, 1994) and a *near-strong Nash equilibrium* (Rozenfeld and Tennenholtz, 2010) as examples of such equilibria.

What is new in this study is the introduction of a new concept of *coalitional equilibria with restricted deviations*, which makes it possible to unify the analysis with the intermediate equilibria. The coalitional equilibrium with restricted deviations is a non-cooperative equilibrium that is stable only against some restricted deviations. The restricted deviations consist of the set of feasible coalitions and feasible deviation strategies for each feasible coalition. They capture the idea that for geographical, legal, or political reasons and so on, not every player can communicate with each other and coalitions that can form are restricted; each feasible coalition faces a self-enforcing problem and its feasible deviation strategies are surely restricted in order for it to execute the deviation. The merit of the

[1] See Quartieri and Shinohara (2015) for examples of games that have multiple pure-strategy Nash equilibria.

coalitional equilibrium with restricted deviations is that we can adequately restate several familiar equilibrium concepts by setting the structure of feasible coalitions and that of feasible deviation strategies, which is formally stated in Proposition 1 below.

We impose an admissible condition on the structure of feasible coalitions, so that deviations by each individual player is possible. We impose two natural self-enforceabilities for deviation strategies, *Nash stability* (NS) and *irreversibility* (IR). NS requires that the deviation strategies of each feasible coalition must be a Nash equilibrium in the game induced by taking players' strategies outside the coalition as fixed. IR requires that the deviation strategies of each feasible coalition must be robust to switching-back options: after the deviation, no member of the coalition switches back to the strategy before deviation, taking the others' strategies as fixed.[2]

We examine how effectively the coalitional equilibria under NS and IR refine Nash equilibria. We first show that under the admissible structure of feasible coalitions, the set of the coalitional equilibria with NS coincides with the set of Nash equilibria in every game with σ-SS and σ-ME. Hence, the coalitional equilibria with NS does not refine the Nash equilibria. We second show that in games with σ-single crossing property, which is stronger than σ-SS, and σ-ME, the set of the coalitional equilibria with IR coincides with the set of Nash equilibria. While there is an example of a game with σ-SS and σ-ME in which the coalitional equilibrium with IR refines the Nash equilibrium, as Example 1 shows, there is a set of games with the same conditions in which Nash equilibria are multiple and the coalitional equilibria with IR does not refine the Nash equilibrium.

We conclude that under NS, which seems to be acceptable as coalitional self-enforceability in non-cooperative games, it is impossible that the coalitional equilibrium provides effective refinements of the Nash equilibrium for games with σ-SS and σ-ME that have multiple Nash equilibria. If we would like to single out some particular Nash equilibria among all the equilibria, we must make the self-enforcing requirement weaker than the NS. The IR is one of the examples. However, it is another problem whether or not we accept the IR or weaker concepts, which do not satisfy the NS, as a self-enforcing requirement because the NS can be considered as a "minimal requirement" for self-enforceability of coalition deviations. Therefore, to refine the Nash-equilibrium analysis through the coalitional equilibria, we must apply self-enforcing conditions, which are mathematically definable but may be unjustifiable as "natural" coalitional behavior in economic meaning.

Related literature

Ichiishi (1981) introduces a *social coalitional equilibrium*, which includes the Nash equilibrium and the core of cooperative games with nonsidepayments as special cases. In Ichiishi's (1981) equilibrium, each feasible coalition faces a restriction of deviation strategies. The feasible deviation strategies of a coalition depends on strategies of players outside the coalition, as ours does. However, Ichiishi (1981) does not consider suitable notions of coalitional self-enforceability, unlike ours. Also, in his equilibrium, the coalition that can be formed is *not* restricted: deviations by any coalition are possible. Zhao (1992) introduces the hybrid solution, which can also express the Nash equilibrium

[2]As we will see later, the set of self-enforcing deviations in w and sCPN equilibria satisfy NS and IR. The set of feasible deviations in near-strong Nash equilibria satisfy IR.

and the core by setting coalition structures appropriately. Laraki (2009) introduces an equilibrium concept called a *coalitional equilibrium*. Like ours, in his equilibrium, the coalition formation is restricted. However, unlike ours, the deviation strategies of each coalition are not restricted. His is equivalent with ours if each feasible coalition can take all joint strategies in our equilibrium. In this sense, ours is more general than Laraki's. Finally, we would like to add that Ichiishi (1981), Zhao (1992), and Laraki (2009) focus on the existence of equilibria, but not on their characterization of it. The coalitional refinements of Nash equilibria have been well studied for games with strategic *complements*. See Milgrom and Roberts (1996), Quartieri (2013), and Shinohara (2019).

2 Preliminaries

2.1 Strategic substitutes and monotonic externalities in σ-interactive games

A strategic-form game is a list $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, in which N is a finite and nonempty set of players and, for each $i \in N$, $S_i \neq \emptyset$ is player i's strategy set and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is player i's payoff function.[3] A subset of N is called a *coalition*. For each coalition $C \subseteq N$ and each strategy profile $s \in \prod_{i\in N} S_i$, the set of strategy profiles for coalition C, $\prod_{i\in C} S_i$, is denoted by S_C. A typical element of S_C is denoted by s_C. Using this notation, we can express $s = (s_C, s_{N\setminus C})$ for each $s \in S_N$. If a coalition is a singleton (that is, $C = \{i\}$ for some $i \in N$), then we simply denote its strategy profile by s_i and its set of strategy profiles S_i. Hereafter, the complement of coalition $\{i\}$ is denoted by $-i$, not $N\setminus\{i\}$, for simplicity.

For the game Γ, the *best response correspondence* of player $i \in N$ is defined as $b_i : S_N \to 2^{S_i}$ such that

$$b_i : s \mapsto \arg\max_{z\in S_i} u_i(z, s_{-i}).$$

The game on which we focus satisfies σ*-interactivity*, which is defined as follows:

Definition 1 A game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a σ-interactive game if and only if

1. $S_i \subseteq \mathbb{R}$ for each $i \in N$ and
2. For each $i \in N$, there exists a function $\sigma_i : S_N \to \mathbb{R}$ such that σ_i is non-decreasing in s_j ($j \neq i$) and constant in s_i; for all $s, \tilde{s} \in S_N$, if $s_i = \tilde{s}_i$ and $\sigma_i(s) = \sigma_i(\tilde{s})$, then $u_i(s) = u_i(\tilde{s})$.

σ-strategic substitutes and σ-single-crossing property exhibit non-increasing properties of the best response function with regard to strategies of the rival players.

Definition 2 A σ-interactive game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ satisfies σ*-interactive strategic substitutes* (σ-SS) if and only if for all $(x, y, i) \in S_N \times S_N \times N$,

$$z_i \in b_i(x),\ w_i \in b_i(y) \text{ and } \sigma_i(x) < \sigma_i(y) \text{ implies } w_i \leq z_i.$$

[3]The model in this study is based on that in Quartieri and Shinohara (2015).

Definition 3 A game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-single crossing property* (σ-SCP) if and only if for all $x, y \in S_N$ and all $i \in N$, if $x_i < y_i$, $\sigma_i(x) < \sigma_i(y)$, and $u_i(x) - u_i(y_i, x_{-i}) \geq 0$, then $u_i(x_i, y_{-i}) - u_i(y) > 0$.

Note that for each game, σ-SCP implies σ-SS, but the converse is not true. A game in Example 1 below satisfies σ-SS, but not σ-SCP.

Definition 4

- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-increasing externalities* (σ-IE) if and only if for all $x, y \in S_N$ and all $i \in N$, if $x_i = y_i$ and $\sigma_i(x) \leq \sigma_i(y)$, then $u_i(x) \leq u_i(y)$.
- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-decreasing externalities* (σ-DE) if and only if $[N, S_N, (-u_i)_{i\in N}]$ is a game with σ-IE.
- A σ-interactive game $\Gamma = [N, S_N, (u_i)_{i\in N}]$ satisfies *σ-monotone externalities* (σ-ME) if and only if Γ satisfies σ-IE or σ-DE.

Our focus is limited to pure-strategies. Quartieri and Shinohara (2015) present several examples of games of economic interest that satisfy σ-SS and σ-ME. They show examples that have multiple pure-strategy Nash equilibria. Hence, the class of our games also possibly includes games with multiple equilibria.

2.2 Preliminary results on coalition-proofness

The Nash equilibrium is defined as usual.

Definition 5 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. A strategy profile $s \in S_N$ is a *Nash equilibrium* for Γ if and only if $s_i \in b_i(s)$ for all $i \in N$. The set of Nash equilibria in Γ is denoted by E_N^Γ.

Pareto domination among strategy profiles are also usual.

Definition 6 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. A strategy profile $s \in S_N$ *strongly Pareto dominates* in Γ a strategy profile $z \in S_N$ if and only if $u_i(z) < u_i(s)$ for all $i \in N$. The *s-efficient subset of* E_N^Γ is the set of Nash equilibria for Γ that are not strongly Pareto dominated in Γ by other Nash equilibria for Γ. The s-efficient subset of E_N^Γ is denoted by sF_N^Γ.

For preparation, we introduce a notion of induced games.

Definition 7 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. Let $C \in 2^N \setminus \{\emptyset\}$, $s \in S_N$, and for all $i \in C$, $\tilde{u}_i : S_C \to \mathbb{R}$, $\tilde{u}_i : z \mapsto u_i(z, s_{-C})$. The *game induced by C at s* is the game $(C, (S_i)_{i\in C}, (\tilde{u}_i)_{i\in C})$ and is denoted by $\Gamma|s_{-C}$.

A *coalition-proof Nash equilibrium*, introduced in Bernheim et al. (1987), is as follows.

Definition 8 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game. If $|N| = 1$, then $s \in S_N$ is an s-coalition-proof Nash equilibrium for Γ if and only if $s \in E_N^\Gamma$. Now assume that $|N| \geq 2$ and that an s-coalition-proof Nash equilibrium has been defined for games with fewer than $|N|$ players. Then,

- $s \in S_N$ is an *s-self-enforcing strategy* for Γ if and only if it is an s-coalition-proof Nash equilibrium for $\Gamma|s_{-C}$ for all nonempty $C \subset N$;

- $s \in S_N$ is an *s-coalition-proof Nash equilibrium* for Γ if and only if it is s-self-enforcing for Γ and there does not exist another s-self-enforcing strategy for Γ that strongly Pareto dominates s in Γ.

The set of s-coalition-proof Nash equilibria in Γ is denoted by E^{Γ}_{sCPN}.

By definition, it is clear that $E^{\Gamma}_{sCPN} \subseteq E^{\Gamma}_N$ and $sF^{\Gamma}_N \subseteq E^{\Gamma}_N$ for all Γ. As pointed out by Bernheim et al. (1987), sF^{Γ}_N and E^{Γ}_{sCPN} are not related by inclusion for some Γ. However, Quartieri and Shinohara (2015) show the equivalence between these three sets in games with σ-SS and σ-ME.

Result 1 (Quartieri and Shinohara, 2015) *Let Γ be a σ-interactive game with σ-strategic substitutes and monotone externalities. Then,*

(1.1) $E^{\Gamma}_N = E^{\Gamma}_{sCPN} = sF^{\Gamma}_N$.

(1.2) *If b_i is single-valued for each $i \in N$, then the set of Nash equilibria and the set of coalition-proof Nash equilibria with weak domination coincide.*[4]

The results suggest that in games with strategic substitutes and monotone externalities, it seems very problematical that the coalition-proof Nash equilibrium refines the set of Nash equilibria when they are multiple. Hence, our question moves to whether some other equilibrium concepts, which are stronger than the coalition-proof Nash equilibrium, refine the Nash equilibrium.

2.3 Coalitional equilibria with restricted deviations

We introduce a new concept, called a *coalitional equilibrium with restricted deviations.* We provide a general notion of restriction of coalition deviations such that the coalition deviations can be restricted to express some earlier equilibrium concepts.

For a game $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$, $\mathcal{C} \subseteq 2^N \backslash \{\emptyset\}$ is defined as a nonempty set of feasible coalitions: only the coalitions in $\mathcal{C}$ can deviate. For each $D \in \mathcal{C}$ and each $s \in S_N$, denote the set of strategies that coalition D can take when deviating from s by R^s_D. Denote $\mathcal{R}_D \equiv (R^s_D)_{s\in S_N}$ for each $D \in \mathcal{C}$ and $\mathcal{R}_{\mathcal{C}} \equiv (\mathcal{R}_D)_{D\in\mathcal{C}}$. We term a pair $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ the set of *feasible deviations.*

Definition 9 Let $\Gamma = \big(N, (S_j)_{j\in N}, (u_j)_{j\in N}\big)$ be a game. Let $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ be the set of feasible deviations. $s \in S_N$ is a $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$*-coalitional equilibrium* in Γ if there do not exist $D \in \mathcal{C}$ and $\tilde{s}_D \in R^s_D$ such that $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ for each $i \in D$. The set of $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$-coalitional equilibrium in Γ is denoted by $E^{\Gamma}_{(\mathcal{C},\mathcal{R}_{\mathcal{C}})}$.

[4] The coalition-proof Nash equilibria with weak domination can be defined as in Definition 8 by replacing strong Pareto dominance with weak Pareto dominance. See, for instance, Shinohara (2005) and Quartieri and Shinohara (2015) for the precise definition.

Next, we introduce a few conditions for the set of feasible deviations. First of all, we introduce the notion of admissibility, which requires that every player can deviate by using every strategy available to it. This requirement seems very natural since each player is assumed to freely choose its strategies in noncooperative games.

Definition 10 $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ is *admissible* if for each $i \in N$ and each $s \in S_N$, $\{i\} \in \mathcal{C}$ and $R_i^s = S_i$.

Henceforth, we assume that admissibility is satisfied.

The *Nash stability* for coalition deviations, defined as follows, seems reasonable under admissibility, because agreed coalition deviations must be immune to the deviation by single members of the coalition under the situation in which every player can take every strategy by admissibility.

Definition 11 Let $\Gamma = \big(N, (S_j)_{j \in N}, (u_j)_{j \in N}\big)$ be a game. $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ satisfies *Nash stability* (NS) if for each $D \in \mathcal{C}$ and each $s \in S_N$,

$$R_D^s \subseteq E_N^{\Gamma|s_{-D}} = \{s'_D \in S_D \,|\, u_i(s'_D, s_{-D}) \geq u_i(s''_i, s'_{D\setminus\{i\}}, s_{-D}) \text{ for each } i \in D \text{ and each } s''_i \in S_i\}.$$

A stability notion weaker than the NS is also introduced as follows:

Definition 12 Let $\Gamma = \big(N, (S_j)_{j \in N}, (u_j)_{j \in N}\big)$ be a game. $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ satisfies *irreversibility* (IR) if for each $D \in \mathcal{C}$ and each $s \in S_N$,

$$R_D^s \subseteq \{s'_D \in S_D \,|\, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\setminus\{i\}}, s_{-D}) \text{ for each } i \in D\}.$$

Denote $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ satisfying NS and that satisfying IR by $(\mathcal{C}, \mathcal{R}_{\mathcal{C}}^{NS})$ and $(\mathcal{C}, \mathcal{R}_{\mathcal{C}}^{IR})$, respectively. These two notions of stability capture the idea of "self-enforceability." That is, although each feasible coalition can freely deviate from certain strategies, no member of feasible coalitions can enforce the other members to take and stick to certain deviation strategies. Hence, for a coalition deviation to be done assuredly, the deviation must be "stable" against any further deviation by proper coalition. NS assumes that once a coalition deviates, then each individual member of the coalition deviates further if he/she has a strategy that improves his/her payoff after the original deviation. Under NS, each feasible coalition can conduct the deviations immune to such further deviations. IR is based on the idea that if a coalition deviates, then each individual member of the coalition has an option to withdraw from the deviation and switch back to the original strategy. Under IR, each feasible coalition deviates in such a way that no member executes such an option. Clearly, if $\mathcal{R}_{\mathcal{C}}$ satisfies NS, then it also satisfies IR. Hence, for each $\mathcal{C}$ and for each Γ, if a strategy profile is a $(\mathcal{C}, \mathcal{R}_{\mathcal{C}}^{IR})$-coalitional equilibrium in Γ, then it is a $(\mathcal{C}, \mathcal{R}_{\mathcal{C}}^{NS})$-coalitional equilibrium in Γ; however, the converse is not true.

Proposition 1 summarizes the relation between the $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$-coalitional equilibrium and several well-known non-cooperative equilibria.

Proposition 1 Let Γ be a game and let $(\mathcal{C}, \mathcal{R}_{\mathcal{C}})$ be an admissible set of feasible deviations.

(1) $E^{\Gamma}_{(\mathcal{C}, \mathcal{R}_{\mathcal{C}})} \subseteq E^{\Gamma}_N$. Further, if $\mathcal{C} = \{\{j\} \,|\, j \in N\}$, then $E^{\Gamma}_{(\mathcal{C}, \mathcal{R}_{\mathcal{C}})} = E^{\Gamma}_N$.

(2) If $C = 2^N \backslash \{\emptyset\}$ and $R^s_D = S_D$ for each $D \in C$ and each $s \in S_N$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *strong Nash equilibrium* (Aumann, 1959).[5]

(3) If $C = \{N, \{j\}_{j \in N}\}$ and $R^s_D = E_N^{\Gamma|s_{-D}}$ for each $D \in C$ and each $s \in S_N$, then $E^{\Gamma}_{(C,\mathcal{R}_C)} = sF^{\Gamma}_N$.

(4) If $C = 2^N \backslash \{\emptyset\}$ and $R^s_D = E_N^{\Gamma|s_{-D}}$ for each $D \in C$ for each $s \in S_N$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *semi-strong Nash equilibrium* in Γ (Kaplan, 1992; Milgrom and Roberts, 1994).[6] Under the same C and R^s_D for each $D \in C$ and each $s \in S_N$, $(C, \mathcal{R}_C)$-coalitional equilibrium is an s-coalition-proof Nash equilibrium in Γ.

(5) If $C = 2^N \backslash \{\emptyset\}$ and $R^s_D = \{s'_D \in S_D | \, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\backslash\{i\}}, s_{-D})$ for each $i \in D\}$ for each $s \in S_N$ and each $D \in C$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is equivalent with the *near-strong Nash equilibrium* (Rozenfeld and Tenneholz, 2010).[7]

Proof of Proposition 1 is in the Appendix.

Remark 1 Milgrom and Roberts (1996) incorporate another notion of restricted coalition formation into the coalition-proof Nash equilibrium. They define a coalition deviation process as a finite sequence of coalitions $\sigma = (C_1, \ldots, C_m)$ such that m is a positive integer and $C_m \subsetneq \cdots \subsetneq C_1 \subseteq N$: this sequence indicates that C_1 can communicate to deviate from a strategy profile; once C_1 has deviated, then C_2 can plan a further deviation from the C_1's deviation, and so on. The set of such sequences, generically denoted by Σ, is called a *coalition communication structure* (CCS). Milgrom and Roberts (1996) impose CCS on some admissibility conditions, which implies that every coalition in the sequences take a Nash equilibrium in the corresponding induced game, and they define a *coalition-proof Nash equilibrium with CCS* along the sequences in Σ, recursively. As in Definition 8, each feasible coalition designated by CCS takes a self-enforcing deviation when deviating. By the definition of the coalition-proof Nash equilibria with CCS, the self-enforcing deviations must be a Nash equilibrium in the corresponding induced game. Thus, if $C = \{D \subseteq N | D$ is the first element of some $\sigma \in \Sigma\}$ and $R^s_D = E_N^{\Gamma|s_{-D}}$ for each $s \in S_N$ and each $D \in C$, then $(C, \mathcal{R}_C)$-coalitional equilibrium is a coalition-proof Nash equilibrium with Σ.[8]

3 Results

The following lemma is a result commonly used in the proof of Propositions 2 and 3.

Lemma 1 Let $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a σ-interactive game with σ-ME. For each $s \in E^{\Gamma}_N$ and each $\tilde{s} \in S_N$, if $u_i(\tilde{s}) > u_i(s)$ for each $i \in N$, then $\sigma_i(s) < \sigma_i(\tilde{s})$ for each $i \in N$ when Γ is a game with σ-IE and $\sigma_i(s) > \sigma_i(\tilde{s})$ for each $i \in N$ when Γ is a game with σ-DE.

[5] A strategy profile $s \in S_N$ is a strong Nash equilibrium if and only if there is no pair $(D, \tilde{s}_D) \in 2^N \backslash \{\emptyset\} \times S_D$ such that $u_i(s) < u_i(\tilde{s}_D, s_{-D})$ for each $i \in D$.

[6] A strategy profile $s \in S_N$ is a semi-strong Nash equilibrium if and only if there is no pair $(D, \tilde{s}_D) \in 2^N \backslash \{\emptyset\} \times E_N^{\Gamma|s_{-D}}$ such that $u_i(s) < u_i(\tilde{s}_D, s_{-D})$ for each $i \in D$.

[7] A strategy profile $s \in S_N$ is a *near-strong Nash equilibrium* if there is no pair $(D, \tilde{s}_D) \in 2^N \backslash \{\emptyset\} \times S_D$ such that for each $i \in D$, $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ and $u_i(\tilde{s}_D, s_{-D}) \geq u_i(\tilde{s}_{D\backslash\{i\}}, s_{-D\cup\{i\}})$.

[8] Shinohara (2010) shows that the set of coalition-proof Nash equilibria with CCS coincides with the entire set of Nash equilibria in games with strategic substitutes and monotone externalities. See Proposition 2 of Shinohara (2010).

Proof. We provide a proof in the case of σ-IE. The case of σ-DE is similar. Suppose that there exists $j \in N$ such that $u_j(\tilde{s}) > u_j(s)$ and $\sigma_j(s) \geq \sigma_j(\tilde{s})$. Since $s \in E^{\Gamma}_N$, then $u_j(s) \geq u_j(\tilde{s}_j, s_{-j})$. Since σ_j is constant in the j-th argument, then $\sigma_j(s) = \sigma_j(\tilde{s}_j, s_{-j}) \geq \sigma_j(\tilde{s})$. By σ-IE, $u_j(\tilde{s}_j, s_{-j}) \geq u_j(\tilde{s})$. Thus, $u_j(\tilde{s}) \leq u_j(s)$, which is a contradiction. ■

Note that this lemma is irrelevant to σ-SS.

3.1 Coalitional equilibria with NS

Proposition 2 Suppose that Γ is a σ-interactive game with σ-SS and σ-ME, C is an admissible set of feasible coalitions, and $\mathcal{R}_C$ are feasible deviations satisfying NS. Then, $E^{\Gamma}_{(C,\mathcal{R}_C)} = E^{\Gamma}_N$.

Proof. Consider games with σ-IE. The proof for the games with σ-DE is similar. By part (1) of Proposition 1, $E^{\Gamma}_{(C,\mathcal{R}_C)} \subseteq E^{\Gamma}_N$. We show the converse. Suppose, to the contrary, that there exists $s \in E^{\Gamma}_N \backslash E^{\Gamma}_{(C,\mathcal{R}_C)}$. Then, $D \in C$ and $\tilde{s}_D \in E^{\Gamma|s_{-D}}_N$ exist such that $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ for each $i \in D$. Since $s \in E^{\Gamma}_N$ and $\tilde{s}_D \in E^{\Gamma|s_{-D}}_N$, then $s_i \in b_i(s)$ and $\tilde{s}_i \in b_i(\tilde{s}_D, s_{-D})$ for each $i \in D$. Applying Lemma 1 to $\Gamma|s_{-D}$ yields $\sigma_i(\tilde{s}_D, s_{-D}) > \sigma_i(s)$ for each $i \in D$. Since σ_i is non-decreasing in all arguments, then the last inequality implies that there exists $i^* \in D$ such that $\tilde{s}_{i^*} > s_{i^*}$. However, by σ-SS, $\sigma_{i^*}(\tilde{s}_D, s_{-D}) > \sigma_{i^*}(s)$, $s_{i^*} \in b_{i^*}(s)$, and $\tilde{s}_{i^*} \in b_{i^*}(\tilde{s}_D, s_{-D})$ imply $\tilde{s}_{i^*} \leq s_{i^*}$, which is a contradiction.

■

By Proposition 2, in games with σ-SS and σ-ME, the $(C, \mathcal{R}^{NS}_C)$-coalitional equilibrium exists whenever the Nash equilibrium does. However, no $(C, \mathcal{R}^{NS}_C)$-coalitional equilibrium refines the set of Nash equilibria. As we see in (4) of Proposition 1, the semi-strong Nash equilibrium, which is stronger than the s-coalition-proof Nash equilibria, can be expressed by a $(C, \mathcal{R}^{NS}_C)$-coalitional equilibrium with some $(C, \mathcal{R}^{NS}_C)$. Even if we use an equilibrium concept that is stronger than the s-coalition-proof Nash equilibrium, if it is based on the NS, then it never refines the Nash equilibrium in games with σ-SS and σ-ME. This result points out the difficulty of refining the Nash equilibria by the equilibrium based on NS.

This result stems from the order structure of the set of Nash equilibria in games with σ-SS. As Quartieri and Shinohara (2015) show in their Theorem 2, in each game Γ with σ-SS, it is impossible that $\sigma_i(s) < \sigma_i(\tilde{s})$ for all $i \in N$ and all distinct $s, \tilde{s} \in E^{\Gamma}_N$. However, by Lemma 1, in each game with σ-IE (resp. σ-DE), s strongly Pareto dominates $\tilde{s}$ only if $\sigma_i(s) < \sigma_i(\tilde{s})$ (resp. $\sigma_i(s) > \sigma_i(\tilde{s})$) for each $i \in N$. These apply to any game induced by any $s' \in S$ and any $D \subseteq N$. Therefore, NS and coalitional profitability are incompatible in games with σ-SS and σ-ME.

3.2 Coalitional equilibria with IR

We examine whether a coalitional equilibrium with IR, which is stronger than that with NS, refines the Nash equilibrium in σ-interactive games with σ-SS and σ-ME. Proposition 3 shows that given C, the coalitional equilibria with IR do not refine the Nash equilibria in a proper subclass of σ-interactive games with σ-SS and σ-ME.

Proposition 3 Suppose that Γ is a σ-interactive game with σ-SCP and σ-ME, C is an admissible set of feasible coalitions, and $\mathcal{R}_C$ represents feasible deviations satisfying IR. Then, $E^{\Gamma}_{(C,\mathcal{R}_C)} = E^{\Gamma}_N$.

Proof. We treat the case of σ-IE. The case of σ-DE is similar. By part (1) of Proposition 1, $E^{\Gamma}_{(C,\mathcal{R}_C)} \subseteq E^{\Gamma}_{N}$. We show the converse. Suppose, to the contrary, that there exists $s \in E^{\Gamma}_{N} \backslash E^{\Gamma}_{(C,\mathcal{R}_C)}$. Then, there exists $D \in C$ and $\tilde{s}_D \in R^s_D$ such that for each $i \in D$, (a) $u_i(\tilde{s}_D, s_{-D}) > u_i(s)$ and (b) $u_i(\tilde{s}_D, s_{-D}) \geq u_i(s_i, \tilde{s}_{D\backslash\{i\}}, s_{-D})$. By (a), applying Lemma 1 to $\Gamma|s_{-D}$ yields $\sigma_i(s) < \sigma_i(\tilde{s}_D, s_{-D})$ for each $i \in D$. By this condition, we have $s_{i^*} < \tilde{s}_{i^*}$ for some $i^* \in D$. Since s is a Nash equilibrium, then $u_{i^*}(s) - u_{i^*}(\tilde{s}_i, s_{-i}) \geq 0$. By the σ-SCP, $s_{i^*} < \tilde{s}_{i^*}$, and $\sigma_{i^*}(s) < \sigma_{i^*}(\tilde{s}_D, s_{-D})$, we reveal that $u_{i^*}(s_i, \tilde{s}_{D\backslash\{i\}}, s_{-D}) - u_{i^*}(\tilde{s}_D, s_{-D}) > 0$, which is a contradiction with (b). ■

The set of σ-interactive games with σ-SCP is a proper subset of the set of games with σ-SS. For example, see Example 1, which provides a game satisfying σ-SS but not σ-SCP. The implication of the result is that in games with σ-SCP and σ-ME, the coalitional equilibrium with IR exists whenever a Nash equilibrium exists. However, the coalitional equilibrium with IR never refines the Nash equilibrium.

However, the following example shows the possibility that the coalitional equilibrium with IR works as a refinement of the Nash equilibrium in games with σ-SS and σ-ME but not σ-SCP.

Example 1 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be such that $N = \{1, 2\}$ and for each $i \in N$, $S_i = [0, 5.6]$ and

$$u_i(s_i, s_j) = \begin{cases} 10 - s_i - s_j & \text{if } s_i \in [0,1] \\ \dfrac{2}{9-s_j}s_i + \dfrac{79 - 18s_j + s_j^2}{9 - s_j} & \text{if } s_i \in [1, \min\{10 - s_j, 5.6\}] \\ 10 - s_j & \text{if } s_i \in \big(\min\{10 - s_j, 5.6\}, 5.6\big] \text{ and } s_j > 4.4 \end{cases}, \tag{1}$$

where $i \neq j$. Suppose that $C = 2^N \backslash \{\emptyset\}$ and $R^s_D = \{s'_D \in S_D |\, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\backslash\{i\}}, s_{-D})$ for each $i \in D\}$ for each $D \in C$ and each $s \in S_N$. We denote a typical graph of u_i when fixing $s_j = z$ in Figure 1.

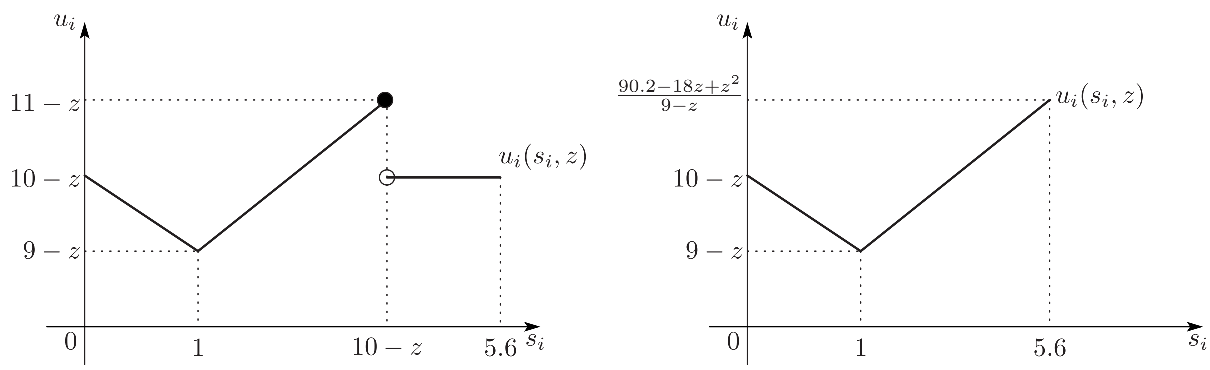

Figure 1: The graph of u_i. The left figure is the case of $z > 4.4$ and the right figure is the case of $z \leq 4.4$.

Fact 1 Let $\sigma_i(s) = s_j$ for each pair $i, j \in N$ such that $i \neq j$ and each $s \in S_N$. This game is then a game with σ-SS and σ-DE, but not σ-SCP.

Fact 2 It follows that $\emptyset \neq E^{\Gamma}_{(C,\mathcal{R}_C)} \subsetneq E^{\Gamma}_{N}$.

Proofs of Facts 1 and 2 are in the Appendix.

By Proposition 2, no coalitional equilibria based on Nash stable coalitional deviations refine the Nash equilibria: hence, the s-coalition-proof Nash equilibria and the semi-strong Nash equilibria are not refinements of Nash equilibria. In addition, since the best response correspondence of each player is singleton-valued, then the set of coalition-proof Nash equilibria under *weak Pareto domination* does not refine the Nash equilibria either (see Result 1-(1.2)).[9] Of course, no strong Nash equilibrium exists. Thus, by this example, we can point out that in games with σ-SS and σ-ME, but not σ-SCP, a coalitional equilibrium with IR may provide a refinement of the Nash equilibrium, although the equilibrium concepts which are frequently used in economics can not refine the Nash equilibrium.[10]

4 Conclusion

Introducing the coalitional equilibrium with restricted deviations, we examine how effectively equilibria based on coalitional stability refine Nash equilibria in σ-interactive games with strategic substitutes and monotone externalities. The coalitional equilibrium with restricted deviations can express several familiar equilibria as special cases by setting feasible coalition deviations appropriately. Thereby, we can provide a unified analysis for the issue.

We impose two stability conditions (NS and IR) on feasible coalition deviations. First, we have shown that the set of the coalitional equilibria with NS coincides with the set of Nash equilibria in every σ-interactive game with σ-SS and σ-ME. Hence, the coalitional equilibria with NS does not refine the Nash equilibria in that game. Second, we have pointed out the possibility that the coalitional equilibrium with IR, which is stronger than the equilibrium with NS, singles out a particular Nash equilibrium from all Nash equilibria in that game. We observe this possibility in σ-interactive games that satisfy σ-SS, but not σ-SCP (see Example 1).

If no member of a coalition can force other members to take certain deviation strategies, then whether the coalition deviation is possible depends on whether it is “self-enforcing”. As discussed previously, requiring the NS on coalition deviations seems reasonable in non-cooperative games because the NS is immune to all single-member deviations of the coalition. Hence, we can consider the NS as the “minimal requirement” for self-enforceability of coalition deviations. On the other hand, the IR is weaker than the NS, and hence it does not satisfy this minimal requirement. If we would like to single out a particular Nash equilibrium from multiple Nash equilibria, we must apply self-enforcing conditions, which are mathematically definable but may be unjustifiable as “natural" coalitional behavior in economic meaning.

Appendix: Proofs

Proof of Proposition 1

(1), (2), (3), and (5) are immediate from the definitions of equilibria.

[9]The coalition-proof Nash equilibrium under weak Pareto domination is defined by replacing strong Pareto dominance of s-coalition-proof Nash equilibria with weak Pareto dominance. See also Corollary 2 in Quartieri and Shinohara (2015).

[10]For further information, all Nash equilibria in this example are *strict* Nash equilibria, which are also trembling-hand-perfect Nash equilibria (Selten, 1975; Okada, 1981). Hence, trembling perfection does not single out a particular Nash equilibrium either.

(4) Let $D \in C$ and let $s \in E^{\Gamma}_{(C,\mathcal{R}_C)}$. First, suppose that $R^s_D = E^{\Gamma|s_{-D}}_N$. By the definition of s-coalition-proof Nash equilibrium, the set of s-self-enforcing deviations of D is a subset of $E^{\Gamma|s_{-D}}_N$. Second, suppose that $R^s_D = \{s'_D \in S_D |\, u_i(s'_D, s_{-D}) \geq u_i(s_i, s'_{D\setminus\{i\}}, s_{-D})$ for each $i \in D\}$. We then note that $E^{\Gamma|s_{-D}}_N \subseteq R^s_D$ and the set of s-self-enforcing deviations of D is a subset of $E^{\Gamma|s_{-D}}_N$. Hence, in any case, $(C, \mathcal{R}_C)$-coalitional equilibrium is robust to the self-enforcing deviations. ■

Proof of Fact 1

First, as a preparation for proof of Fact 1, we show Claim 1.

Claim 1 If $S_i = [0, 5.6]$ for each $i \in N$, then $10 - s_i - s_j$, $10 - s_j$, and $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ are decreasing in s_j.

Proof of Claim 1. Clearly, $10 - s_i - s_j$ and $10 - s_j$ are decreasing in s_j. Differentiating $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ in s_j, we have

$$\frac{\partial}{\partial s_j}\left(\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}\right) = \frac{2s_i - s_j^2 + 18s_j - 83}{(9-s_j)^2}.$$

Since $S_i = S_j = [0, 5.6]$, then $2s_i - s_j^2 + 18s_j - 83$ is maximized at $(s_i, s_j) = (5.6, 5.6)$ and the maximum value is -2.36. Thus, $\frac{2}{9-s_j}s_i + \frac{79-18s_j+s_j^2}{9-s_j}$ is also decreasing in s_j. **(End of Proof of Claim 1)**

We first verify that this game satisfies σ-DE. Let $s_i = x \in S_i$ and let $s'_j, s''_j \in S_j$ be such that $s''_j < s'_j$. We show that $u_i(x, s''_j) \geq u_i(x, s'_j)$. Note that $1 < \min\{10 - s'_j, 5.6\} \leq \min\{10 - s''_j, 5.6\}$ and the last inequality holds with equality if $s'_j \leq 4.4$.

By Claim 1, if $x \in [0, 1] \cup [1, \min\{10 - s'_j, 5.6\}] \cup (\min\{10 - s''_j, 5.6\}, 5.6]$, then $u_i(x, s'_j) < u_j(x, s''_j)$ because

$$u_i(x,z) = \begin{cases} 10 - x - z & \text{if } x \in [0,1] \\ \frac{2}{9-z}x + \frac{79-18z+z^2}{9-z} & \text{if } x \in [1, \min\{10 - s'_j, 5.6\}] \\ 10 - z & \text{if } x \in (\min\{10 - s''_j, 5.6\}, 5.6] \end{cases}$$

for each $z \in \{s'_j, s''_j\}$. If $x \in \left(\min\{10 - s'_j, 5.6\},\ \min\{10 - s''_j, 5.6\}\right]$, then $u_i(x, s'_j) = 10 - s'_j$ and $u_j(x, s''_j) = \frac{2}{9-s''_j}x + \frac{79-18s''_j+(s''_j)^2}{9-s''_j}$. Denote $s'_j = s''_j + d$, where $d > 0$. Then, we have

$$\begin{aligned} u_j(x, s''_j) - u_j(x, s'_j) &= \frac{-11 + s''_j + 2x + d(9 - s''_j)}{9 - s''_j} \\ &> \frac{-11 + s''_j + 2\min\{10 - s'_j, 5.6\} + d(9 - s''_j)}{9 - s''_j} \\ &= \begin{cases} \frac{9 - s''_j + d(7 - s''_j)}{9 - s''_j} > 0 & \text{if } \min\{10 - s'_j, 5.6\} = 10 - s'_j \\ \frac{0.2 + s''_j + d(9 - s''_j)}{9 - s''_j} > 0 & \text{otherwise} \end{cases} \end{aligned}$$

because $s''_j \leq 5.6$. In conclusion, this game satisfies σ-DE.

We secondly verify that this game satisfies σ-SS. Let $i, j \in N$ be such that $i \neq j$ and $s \in S_N$. First, if $s_j \in (4.4, 5.6]$, then $u_i(s)$ is maximized at $s_i = 10 - s_j$ as we can see in Figure 1. Second, if $s_j \in [0, 4.4]$, then $u_i(s)$ is *locally* maximized at $s_i = 0,\ 5.6$ and $u_i(5.6, s_j) - u_i(0, s_j) = \frac{0.2+s_j}{9-s_j} > 0$. Therefore,

$$b_i(s) = \begin{cases} \{10 - s_j\} & \text{if } s_j \in (4.4, 5.6] \\ \{5.6\} & \text{otherwise} \end{cases}. \tag{2}$$

Clearly, this is a game with σ-SS.

We can also verify that this is not a game with σ-SCP. We have that $u_1(5.2, 4.5) = u_1(5.4, 4.5) = 5.5$ and $u_1(5.2, 5) = u_1(5.4, 5) = 5$; hence, $u_1(5.2, 4.5) - u_1(5.4, 4.5) = u_1(5.2, 5) - u_1(5.4, 5) = 0$, which implies that this game does not satisfy σ-SS. ■

Proof of Fact 2

By (2),

$$E_N^\Gamma = \{(s_1, s_2) : s_1 + s_2 = 10 \text{ and } 4.4 \leq s_i \leq 5.6 \text{ for each } i \in N\}.$$

First, we verify $e^* = (5, 5) \in E_N^\Gamma \backslash E_{(C,\mathcal{R}_C)}^\Gamma$. The payoff to all $i \in N$ at e^* is $u_i(e^*) = 6$. If the two players deviate from e^* to $e = (0, 0)$, then $u_i(e) = 10$ for all $i \in N$. If player i switches back to $e_i^* = 5$ given e_j for $j \neq i$, then $u_i(e_i^*, e_j) = \frac{89}{9}$. Therefore, no player i switches back to the original strategy e_i^*.

Second, we verify that $e^{**} = (4.4, 5.6) \in E_{(C,\mathcal{R}_C)}^\Gamma$. At e^{**}, $u_1(e^{**}) = 5.4$ and $u_2(e^{**}) = 6.6$. Let $\tilde{s} \in S_N$ be deviating strategies from e^{**} such that $u_i(\tilde{s}) > u_i(e^{**})$ for each $i \in N$. Since this is a game with σ-DE, then $e_i^{**} > \tilde{s}_i$ for each $i \in N$ by Lemma 1. We then have

$$\tilde{s}_i < e_i^{**} = 10 - e_j^{**} < 10 - \tilde{s}_j \text{ for all } i, j \in N \text{ such that } i \neq j. \tag{3}$$

Claim 2 If there exist distinct $i, j \in N$ such that $\tilde{s}_i \in \left[1, \min\{10 - \tilde{s}_j, 5.6\}\right]$, then $u_i(e_i^{**}, \tilde{s}_j) > u_i(\tilde{s})$.

Proof of Claim 2. By (3), $e_i^{**} \in [1, \min\{10 - \tilde{s}_j, 5.6\}]$. Hence, for each $x \in \{e_i^{**}, \tilde{s}_i\}$,

$$u_i(x, \tilde{s}_j) = \frac{1}{9 - \tilde{s}_j}\left(2x + 79 - 18\tilde{s}_j + (\tilde{s}_j)^2\right)$$

and $e_i^{**} > \tilde{s}_i$ implies $u_i(e_i^{**}, \tilde{s}_j) > u_j(\tilde{s})$. **(End of Proof of Claim 2)**

Claim 3 If there exist distinct $i, j \in N$ such that $\tilde{s}_i \in \left(\min\{10 - \tilde{s}_j, 5.6\}, 5.6\right]$, then $u_i(e_i^{**}, \tilde{s}_j) > u_i(\tilde{s})$.

Proof of Claim 3. Since $e_1^{**} = 4.4 > \tilde{s}_1$, then it is impossible that $i = 2$ and $j = 1$. (If $i = 2$ and $j = 1$, then $\left(\min\{10 - \tilde{s}_j, 5.6\}, 5.6\right]$ is empty.) We consider the case of $i = 1$ and $j = 2$. In this case, note that $\tilde{s}_2 \geq 4.4$. Since

$\tilde{s}_2 \le 5.6$, then $\min\{10 - \tilde{s}_2, 5.6\} \ge 4.4$. Since $e_1^{**} = 4.4$, then $e_1^{**} \in [1, \min\{10 - \tilde{s}_2, 5.6\}]$. Hence,

$$\begin{aligned} u_1(\tilde{s}) - u_1(e_1^{**}, \tilde{s}_2) &= 10 - \tilde{s}_2 - \left(\frac{87.8 - 18\tilde{s}_2 + (\tilde{s}_2)^2}{9 - \tilde{s}_2}\right) \\ &= \frac{2.2 - \tilde{s}_2}{9 - \tilde{s}_2} < 0. \end{aligned}$$

(End of Proof of Claim 3)

Claim 4 If $\tilde{s} \in [0, 1]^2$, then $u_2(\tilde{s}_1, e_2^{**}) > u_2(\tilde{s})$.

Proof of Claim 4. By (1), since $\tilde{s} \in [0, 1]^2$, then $u_i(\tilde{s}) \in [8, 10]$ for each $i \in N$. We have

$$\begin{aligned} u_2(\tilde{s}) - u_2(\tilde{s}_1, e_2^{**}) &= 10 - \tilde{s}_1 - \tilde{s}_2 - \left(\frac{90.2 - 18\tilde{s}_1 + (\tilde{s}_1)^2}{9 - \tilde{s}_1}\right) \\ &= -\frac{0.2 + \tilde{s}_1 + \tilde{s}_2(9 - \tilde{s}_1)}{9 - \tilde{s}_1} < 0 \end{aligned}$$

because $\tilde{s}_1 \le 1$. **(End of Proof of Claim 4)**

By Claims 2 to 4, for each improving deviation $\tilde{s}$, there is at least one player $k \in N$ that switches back to e_k^{**}. Therefore, $e^{**} \in E^{\Gamma}_{(C,\mathcal{R}_C)}$.

In conclusion, $\emptyset \neq E^{\Gamma}_{(C,\mathcal{R}_C)} \subsetneq E^{\Gamma}_N$. ∎

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