Coalition-proof Nash Equilibrium of Aggregative Games

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Abstract

We clarify the properties of a coalition-proof Nash equilibrium in an aggregative game with *monotone externality* and *strategic substitution*. In this aggregative game, since no coalition can deviate from any Nash equilibrium in self-enforcing and payoff-improving ways, the sets of Nash equilibria and coalition-proof Nash equilibria coincide. We clarify that the interesting properties of the coalition-proof Nash equilibrium observed by Yi (1999) and Shinohara (2005) stem from the equivalence between these two equilibria. Other interesting properties are also based on this equivalence. We examine the relationship between coalition-proofness and restricted coalition formation and between coalition-proofness and the iterative elimination of weakly dominated strategies.

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1 Introduction

We clarify the properties of a coalition-proof Nash equilibrium in an aggregative game with *monotone externality* and *strategic substitution*.

The coalition-proof Nash equilibrium introduced by Bernheim et al. (1987) is a refinement of the Nash equilibrium. The coalition-proof Nash equilibrium is immune

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to any self-enforcing coalitional deviation. The self-enforcing deviation of a coalition is a deviation from which no proper subcoalition of the coalition can deviate profitably using its self-enforcing deviation. Hence, in order to define the self-enforcing deviation of a coalition, the self-enforcing deviation of any proper subset of the coalition must be defined. Due to the recursive nature of coalition-proof Nash equilibria, it is not easy to study the properties of the equilibrium.

We focus on an aggregative game with *monotone externality* and *strategic substitution*. The aggregative game is such that the strategy set of any player is a subset of the real line and the payoff of any player depends on his/her strategy and on the sum of the strategies of the other players. *Monotone externality* requires that a switch in a player's strategies changes the payoffs of all the other players in the same direction. *Strategic substitution* means that an incentive to any player to reduce his/her strategy is preserved if the sum of the other players' strategies increases.

In an aggregative game with these conditions, the coalition-proof Nash equilibrium has interesting properties. The set of coalition-proof Nash equilibria coincides with the (weakly) Pareto-efficient frontier of the set of Nash equilibria (Yi, 1999). The set of coalition-proof Nash equilibria based on weak payoff dominance is a subset of the set of coalition-proof equilibria based on strict payoff dominance (Shinohara, 2005). These properties are not satisfied in general (Bernheim et al. 1987; Konishi et al. 1999).

First, we show that in the aggregative game, no group of players can deviate from any Nash equilibrium in such a way that all the players are made better off by using their self-enforcing deviations. Thus, any Nash equilibrium is coalition-proof and the set of Nash equilibria *itself* coincides with the set of coalition-proof Nash equilibria. The equivalence between these two equilibria implies the results of Yi (1999) and Shinohara (2005). Therefore, we can say that this equivalence is the fundamental mechanism for the interesting properties that they report.

On the basis of this equivalence, we can derive the other interesting properties of the coalition-proof Nash equilibrium.

Second, we address how the coalition-proof Nash equilibrium changes when the coalition formation is restricted. The original coalition-proof Nash equilibrium assumes that all possible coalitions can form and deviate. However, in the real world, not all coalitions form for some reason. For example, the coordination of many players is very costly and it is difficult to form a large coalition; some individuals are unable to form a coalition because of geographical compulsions; and firms are prohibited to form a cartel by law. A *coalition communication structure* (CCS), introduced by Milgrom and Roberts (1996), captures the idea of restricted coalition formation. A CCS represents which coalition is feasible. A coalition-proof Nash equilibrium under a CCS is stable against self-enforcing deviations of the coalitions designated by the CCS. The coalition-proof Nash equilibria are generally different under distinct CCSs. However, by imposing a natural restriction on the CCS, we show that any Nash equilibrium is coalition-proof under any CCS and that the sets of coalition-proof Nash equilibria are the same under any CCS if an aggregative game satisfies *monotone externality* and *strategic substitution*.

Third, we examine the relationship between the coalition-proof Nash equilibrium and the iterative elimination of weakly dominated strategies. We show that in our games, if each player's strategy set is finite, (a) any pure-strategy Nash equilibrium that consists of serially undominated strategies in the sense of weak domination is a coalition-proof Nash equilibrium and (b) the serially undominated Nash equilibria do not Pareto-dominate each other. We also find that these statements do not necessarily hold true if an aggregative game does not satisfy one of the conditions of *monotone externality* and *strategic substitution*.

The second result shows that the effect of CCS on coalition-proofness when a game satisfies *strategic substitution* is different from that when a game satisfies *strategic complementarity*. Milgrom and Roberts (1996) show that in a game with *strategic complementarity* and *monotone externality*, some, but not all Nash equilibria are always coalition-proof for each CCS.

The third result shows that the relationship between the coalition-proofness and the iterative elimination of weakly dominated strategies is different from that between the coalition-proofness and the iterative elimination of strictly dominated strategies. The relationship of the equilibrium with the iterative elimination of strictly dominated strategies has already been examined by Moreno and Wooders (1996) and Milgrom and Roberts (1996). Moreno and Wooders (1996) investigate a game with finite strategy sets and show that if there exists a profile of serially undominated strategies that Pareto-dominates the other serially undominated strategies, it is a coalition-proof Nash equilibrium. In contrast, when the iterative elimination of weakly dominated strategies is adopted, a Pareto-superior serially undominated Nash equilibrium is not coalition-proof, as demonstrated by our examples. Hence, some conditions need to be imposed on a game so that a result similar to that of the earlier study holds. Our result provides such conditions for an aggregative game.

Our results also provide a sufficient condition under which there is a coalition-proof Nash equilibrium consisting of undominated strategies. The undominated strategy is not weakly dominated by any other strategy. Obviously, the surviving coalition-proof Nash equilibrium consists of undominated strategies. As Peleg (1997) points out, a coalition-proof Nash equilibrium may consist of weakly dominated strategies. He also proves that almost all dominant-strategy equilibria are coalition-proof; thus, such equilibria consist of undominated strategies. In our class of games, a dominant-strategy equilibrium does not necessarily exist.

2 Model

We consider a strategic game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. For each $i \in N$, let X_i be the set of strategies of player i such that $X_i \subseteq \mathbb{R}$ and let $u_i\colon \prod_{j\in N} X_j \to \mathbb{R}$ be the payoff function of player i. Let $S \subseteq N$ be a coalition. Let $X_S \equiv \prod_{i\in S} X_i$. Denote $x_S \equiv (x_i)_{i\in S} \in X_S$. The complement of S is denoted by $-S$. For notational simplicity, denote $X \equiv \prod_{j\in N} X_j$, $x \equiv (x_j)_{j\in N} \in X$, $X_{-i} \equiv X_{-\{i\}}$, and $x_{-i} \equiv x_{-\{i\}}$ for each $i \in N$.

We focus on an *aggregative game*. In this game, the payoff of any player depends on his/her strategy and on the sum of the strategies of the other players.

Definition 1 A game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$ is an *aggregative game* if $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$ for each $i \in N$, each $x_i \in X_i$, and each pair x_{-i}, $x'_{-i} \in X_{-i}$ such that $\sum_{j\neq i} x_j = \sum_{j\neq i} x'_j$.

We focus on the case in which any player chooses only pure strategies.

The (pure-strategy) Nash equilibrium is defined as usual. Let $NE(G)$ be the set of (pure-strategy) Nash equilibria in G.

To define a coalition-proof Nash equilibrium, we introduce a *restricted game*. The restricted game is defined for each coalition and for each profile of strategies outside the coalition. For each $S \subseteq N$ and each $x_{-S} \in X_{-S}$, the game restricted by x_{-S} is denoted by $G|x_{-S} = [S, (X_i)_{i\in S}, (\widetilde{u}_i)_{i\in S}]$, in which S is the set of players, X_i is the set of strategies for $i \in S$, and $\widetilde{u}_i\colon X_S \to \mathbb{R}$ is the payoff function of $i \in S$ which is defined as $\widetilde{u}_i(x_S) = u_i(x_S, x_{-S})$ for each $x_S \in X_S$.

Definition 2 A *coalition-proof Nash equilibrium* $x \in X$ is defined inductively with respect to the number of members in coalitions:

1. For each $i \in N$, x_i is a coalition-proof Nash equilibrium of $G|x_{-i}$ if $x_i \in \arg\max_{x'_i \in X_i} u_i(x'_i, x_{-i})$.
2. Let S be a coalition with $\#S \geq 2$. Assume that the coalition-proof Nash equilibria have been defined for each proper subset of S. Then, x_S is a coalition-proof Nash equilibrium of $G|x_{-S}$ if
   (a) x_S is a *self-enforcing strategy profile* of $G|x_{-S}$; x_T is a coalition-proof Nash equilibrium of $G|x_{-T}$ for each $T \subsetneq S$ and
   (b) there is no self-enforcing strategy profile y_S of $G|x_{-S}$ such that $u_i(y_S, x_{-S}) > u_i(x_S, x_{-S})$ for each $i \in S$.

The set of coalition-proof Nash equilibria in G is denoted by $CPNE(G)$. For each $S \subseteq N$ and each $x_{-S} \in X_{-S}$, the set of coalition-proof Nash equilibria in the restricted game $G|x_{-S}$ is also denoted by $CPNE(G|x_{-S})$. Similarly, the set of Nash equilibria in the restricted game $G|x_{-S}$ is denoted by $NE(G|x_{-S})$.

In the coalition-proof Nash equilibrium of G, no proper coalition of N can deviate in such a way that the coalition uses the coalition-proof Nash equilibrium of its corresponding restricted game and all members of the coalition are made better off. Clearly, $CPNE(G|x_{-S}) \subseteq NE(G|x_{-S})$ for each non-empty $S \subseteq N$ and each $x_{-S} \in X_{-S}$.

3 Properties of Coalition-proof Nash Equilibrium

Definition 3 A game G satisfies *monotone externality* if either *positive externality* or *negative externality* is satisfied:

Positive externality. For each $i \in N$, for each $x_i \in X_i$, and for each pair x_{-i}, $\widehat{x}_{-i} \in X_{-i}$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \geq u_i(x_i, \widehat{x}_{-i})$.

Negative externality. For each $i \in N$, for each $x_i \in X_i$, and for each pair x_{-i}, $\widehat{x}_{-i} \in X_{-i}$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \leq u_i(x_i, \widehat{x}_{-i})$.

The condition of *monotone externality* requires that the payoff to player i changes monotonically with respect to the strategies of players other than i. The Cournot competition game is an example that satisfies *negative externality*; the voluntary provision game of a pure public good is an example that satisfies *positive externality.*

Definition 4 A game G satisfies *strategic substitution* if for each $i \in N$, each pair $x_i,\ x'_i$ with $x_i > x'_i$, and each pair $x_{-i},\ x'_{-i}$ with $\sum_{j \neq i} x_j > \sum_{j \neq i} x'_j$, if $u_i(x'_i, x'_{-i}) - u_i(x_i, x'_{-i}) \geq 0$, then $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) > 0$.

Here is an interpretation for the *strategic substitution.* Consider a situation in which player i does not have an incentive to choose x_i instead of x'_i when the other players choose x'_{-i}. Even if the other players increase their strategies from x'_{-i}, i does not have such an incentive.[1] *Strategic substitution* is satisfied by many games such as the Cournot competition game and the voluntary provision game of a pure public good.[2]

3.1 Equivalence between Nash Equilibria and Coalition-proof Nash Equilibria

The property presented in Lemma 1 is fundamental to clarify the characteristics of a coalition-proof Nash equilibrium.

Lemma 1 Suppose that an aggregative game G satisfies *monotone externality* and *strategic substitution.* For each $x \in NE(G)$, each non-empty $S \subseteq N$, and each $\widetilde{x}_S \in X_S$, if $u_i(\widetilde{x}_S, x_{-S}) > u_i(x)$ for each $i \in S$, $\widetilde{x}_S$ is not a Nash equilibrium of $G|x_{-S}$.

Proof. We provide the proof in a case in which *positive externality* is satisfied. Similarly, we can show the statement in the case of *negative externality.*

By the definition of Nash equilibria and *positive externality*, if $\sum_{j \neq S \setminus \{i\}} \widetilde{x}_j \leq \sum_{j \neq S \setminus \{i\}} x_j$ for some $i \in S$, $u_i(x) \geq u_i(\widetilde{x}_i, x_{-i}) \geq u_i(\widetilde{x}_i, \widetilde{x}_{S \setminus \{i\}}, x_{-S})$. This contradicts $u_i(\widetilde{x}_S, x_{-S}) > u_i(x)$. Therefore, $\sum_{j \neq S \setminus \{i\}} \widetilde{x}_j > \sum_{j \neq S \setminus \{i\}} x_j$ for each $i \in S$.

By $\sum_{j \neq S \setminus \{i\}} \widetilde{x}_j > \sum_{j \neq S \setminus \{i\}} x_j$ for each $i \in S$, we have $\sum_{j \in S} \widetilde{x}_j > \sum_{j \in S} x_j$. Hence, there exists $k \in S$ such that $\widetilde{x}_k > x_k$. Since $\widetilde{x}_k > x_k$, $\sum_{j \neq S \setminus \{k\}} \widetilde{x}_j > \sum_{j \neq S \setminus \{k\}} x_j$, and $u_k(x_k, x_{-k}) - u_k(\widetilde{x}_k, x_{-k}) \geq 0$, we have $u_k(x_k, \widetilde{x}_{S \setminus \{k\}}, x_{-S}) - u_k(\widetilde{x}_k, \widetilde{x}_{S \setminus \{k\}}, x_{-S}) > 0$ by *strategic substitution.* Hence, $\widetilde{x}_S \notin NE(G|x_{-S})$. ■

[1] Strictly speaking, *strategic substitution* in Definition 4 is weaker than that defined by Yi (1999), which is defined as follows: for each $i \in N$, each pair x_i, x'_i, and each pair x_{-i}, x'_{-i}, if $x_i > x'_i$ and $\sum_{j \neq i} x_j > \sum_{j \neq i} x'_j$, then $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i}) > u_i(x'_i, x'_{-i}) - u_i(x_i, x'_{-i})$. While Yi's (1999) condition implies ours, the converse is not true. Unlike ours, Yi's (1999) condition requires that $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i})$ is increasing in x_{-i} for $x_i > x'_i$. This difference does not matter when proving our results.

[2] The other examples that satisfy these two conditions are provided by Yi (1999).

Lemma 1 says that no coalition, including the grand coalition, can deviate from any Nash equilibrium in such a way that any member of the coalition is made better off and the coalitional deviation is self-enforcing. Thus, the set of Nash equilibria *itself* coincides with that of coalition-proof Nash equilibria.

Proposition 1 Suppose that G satisfies *monotone externality* and *strategic substitution*. The sets of Nash equilibria and coalition-proof Nash equilibria coincide.

Proof. Clearly, $CPNE(G) \subseteq NE(G)$. By Lemma 1, no coalition can deviate from any point in $NE(G)$ in self-enforcing and payoff-improving ways. Thus, $NE(G) \subseteq CPNE(G)$. ■

The results of Yi (1999) and Shinohara (2005) arrives immediately from Proposition 1. By definition, no coalition-proof Nash equilibrium is Pareto-dominated by any other coalition-proof Nash equilibrium. Since $NE(G) = CPNE(G)$ by Proposition 1, $NE(G)$ coincides with the Pareto-efficient frontier of $NE(G)$. Therefore, $CPNE(G)$ coincides with the Pareto-efficient frontier of $NE(G)$.

There are two definitions of coalition-proof Nash equilibria: a coalition-proof Nash equilibrium based on *weak payoff dominance*,[3] $CPNE^{\succsim}$ for short, and that based on *strict payoff dominance*, $CPNE^{\succ}$ for short. The $CPNE^{\succ}$ is defined in Definition 2. Since the $CPNE^{\succsim}$ is a Nash equilibrium and the sets of $CPNE^{\succ}$ and Nash equilibria coincide, the set of $CPNE^{\succsim}$ is included in the set of $CPNE^{\succ}$. We can conclude that the properties observed in earlier studies stem from the equivalence between the Nash equilibrium and the coalition-proof Nash equilibrium.

Corollary 1 Suppose that an aggregative game satisfies *monotone externality* and *strategic substitution.* (a) The set of coalition-proof Nash equilibria and the Pareto-efficient frontier of the set of Nash equilibria coincide (Yi, 1999). (b) The set of $CPNE^{\succ}$ is a subset of that of $CPNE^{\succsim}$ (Shinohara, 2005).

Remark 1 We consider a case in which the players can use mixed strategies. Even in this case, $NE(G) = CPNE(G)$ if X_i is a convex set and $u_i(x_i, x_{-i})$ is strictly concave in x_i for each $i \in N$. In the game G with convex strategy sets and concave payoff functions, $NE(G)$ only consists of the pure strategies. For a deviation of a

[3] Let $S \subseteq N$ and $x_{-S} \in X_{-S}$. Profile $x_S \in X_S$ weakly payoff dominates $x'_S \in X_S$ for S at x_{-S} if $u_i(x_S, x_{-S}) \geq u_i(x'_S, x_{-S})$ for each $i \in S$ with strict inequality for at least one $i \in S$.

coalition to be self-enforcing, any player in the coalition uses a pure strategy through this deviation. Therefore, $NE(G) = CPNE(G)$.

In the following subsections, we prove that a coalition-proof Nash equilibrium satisfies the two distinct properties that have not been observed by earlier studies. The statements are made based on Proposition 1 and Lemma 1.

3.2 Coalition-proof Nash Equilibrium under Restricted Coalition Formation

We examine how the coalition-proof Nash equilibrium changes under different restrictions on coalition formation in an aggregative game with *monotone externality* and *strategic substitution*.

A *coalition communication structure* (CCS), introduced by Milgrom and Roberts (1996), captures the idea of restricted coalition formation. It represents which coalitions can communicate to plan deviation. Let $\sigma = (S_1, \ldots, S_m)$ ($m \geq 1$ and $m \in \mathbb{Z}_+$) be a finite sequence of coalitions such that $S_1 \subseteq N$ and $S_{r+1} \subsetneq S_r$ for each $r \in \{1, \ldots, m-1\}$. Let Σ be a set of such sequences, called a CCS. Given Σ, if $\sigma = (S_1, \ldots, S_m) \in \Sigma$, S_1 can communicate to deviate from a strategy profile; once S_1 has deviated, S_2 can plan a further deviation from S_1's deviation, and so on. For each $S \subseteq N$, (S) denotes a sequence such that S is the only element. For notational simplicity, denote $(\{i\})$ by (i) for each $i \in N$. For each pair $\sigma = (S_1, \ldots, S_m)$, $\sigma' = (T_1, \ldots, T_r)$, $(\sigma, \sigma') \equiv (S_1, \ldots, S_m, T_1, \ldots, T_r)$. A sequence $\underline{\sigma}$ is *initial in* Σ if $(\underline{\sigma}, \sigma') \in \Sigma$ for some σ'. Let $\underline{\sigma}$ be initial in Σ. A *CCS induced by* $\underline{\sigma}$, denoted by $\Sigma(\underline{\sigma})$, is the set of all sequences σ' such that $(\underline{\sigma}, \sigma') \in \Sigma$. Once the coalitions in $\underline{\sigma}$ have deviated, the opportunities for further deviations are denoted by $\Sigma(\underline{\sigma})$. We adopt a convention that $\Sigma(\sigma) = \Sigma$ if $\sigma = \emptyset$. Let (G, Σ) be the game with Σ. Let $(G, \Sigma(\underline{\sigma}))$ be the game with the induced CCS. Similarly, we can define the initial sequence in $\Sigma(\underline{\sigma})$.

Naturally, we assume that any player can freely deviate at any point of the sequences: $(i) \in \Sigma$ for each $i \in N$. For each $\sigma = (S_1, \ldots, S_m)$ such that $m \geq 1$ and $\#S_m \geq 2$, if σ is initial in Σ, then $(\sigma, i) \in \Sigma$ for each $i \in S_m$.

Definition 5 Let Σ be a CCS. A *coalition-proof Nash equilibrium in* (G, Σ) $x \in X$ is defined along the sequences in Σ:

1. For each $i \in N$, x_i is a coalition-proof Nash equilibrium for i at x_{-i} if $x_i \in \arg\max_{x'_i \in X_i} u_i(x'_i, x_{-i})$.

2. Let $S \subseteq N$ be such that $\#S \geq 2$ and (S) is initial in $\Sigma(\sigma)$ for some σ. Assume that the coalition-proof Nash equilibrium for T at x_{-T} has been defined for each $T \subsetneq S$ such that (T) is initial in $\Sigma(\sigma, S)$. Then, x_S is *self-enforcing for S at x_{-S}* if x_T is coalition-proof for each $T \subsetneq S$ at x_{-T} such that (T) is initial in $\Sigma(\sigma, S)$. Profile x_S is a *coalition-proof Nash equilibrium for S at x_{-S}* if there is no self-enforcing $y_S \in X_S$ such that $u_i(y_S, x_{-S}) > u_i(x_S, x_{-S})$ for each $i \in S$.

Denote by $CPNE(G, \Sigma)$ the set of coalition-proof Nash equilibria in (G, Σ). Note that $CPNE(G, \Sigma) \subseteq NE(G)$ for each Σ. Let $\overline{\Sigma}$ be the CCS that consists of all possible decreasing sequences of the subsets of N. Then, $CPNE(G, \overline{\Sigma}) = CPNE(G)$. Let $\underline{\Sigma} = \{\sigma | \, \sigma = (i) \text{ for some } i \in N\}$. Then, $CPNE(G, \underline{\Sigma}) = NE(G)$. It is noteworthy that $CPNE(G, \Sigma)$ and $CPNE(G, \Sigma')$ are not related by inclusion for some Σ and Σ'. For instance, let Σ be such that $\sigma \in \Sigma$ if and only if $\sigma = (i)$ or $\sigma = (N, i)$ for some $i \in N$. Then, $CPNE(G, \Sigma)$ coincides with the Pareto-efficient frontier of the set of Nash equilibria; $CPNE(G, \overline{\Sigma})$ and $CPNE(G, \Sigma)$ do not necessarily intersect.

Proposition 2 Suppose that an aggregative game satisfies *monotone externality* and *strategic substitution*. Then, (a) $CPNE(G, \Sigma) = NE(G) = CPNE(G)$ for each Σ and (b) $CPNE(G, \Sigma) = CPNE(G, \Sigma')$ for each pair Σ, Σ'.

Proof. By Proposition 1, $CPNE(G, \Sigma) \subseteq NE(G) = CPNE(G)$ for each Σ. We show that $NE(G) \subseteq CPNE(G, \Sigma)$. Suppose that this is not the case. Let $x \in NE(G) \backslash CPNE(G, \Sigma)$. Then, $S \subseteq N$ and $\widetilde{x}_S \in X_S$ exist such that (S) is initial in Σ, $\widetilde{x}_S$ is self-enforcing for S, and $u_i(\widetilde{x}_S, x_{-S}) > u_i(x)$ for each $i \in S$. Since $\widetilde{x}_S$ is self-enforcing for S at x_{-S}, $\widetilde{x}_S \in NE(G|x_{-S})$. However, by Lemma 1, $\widetilde{x}_S \notin NE(G|x_{-S})$. This is a contradiction. It is immediate from (a) that (b) holds. ■

By Proposition 2, the possibility of communication does not affect coalition-proof outcomes in an aggregative game with the two conditions. For example, consider the Cournot oligopoly game, which is included in our class of games. Consider a situation in which firms form a cartel and coordinate their quantities, but only small cartels can be formed because coordinating many firms is difficult and costly. Our results imply that the equilibrium outcomes against self-enforcing cartel behavior in this situation are the same as those in the situation in which all possible coalitions are feasible. Thus, in the aggregative game, the self-enforcing behavior of cartels leads to the same outcomes irrespective of how many players can coordinate their strategies together.

Milgrom and Roberts (1996) examine a coalition-proof Nash equilibrium of a game with *strategic complementarity*.[4] The game with *strategic complementarity* has the largest and smallest elements in a serially undominated set in any order.[5] They are the largest and the smallest Nash equilibria, respectively (Milgrom and Shannon, 1994). Milgrom and Roberts (1996) show that if the game satisfies the *positive externality*, only the largest Nash equilibrium is coalition-proof. They also show that if the game satisfies *negative externality*, only the smallest Nash equilibrium is coalition-proof. Even if a CCS is introduced, the largest or smallest Nash equilibrium is coalition-proof depending on whether the externality is positive or negative. Their results are demonstrated by the following example.

Example 1 Let G denote the game depicted in Table 1. We assume that $A_1 > A_2$ and $B_1 > B_2$. *Strategic complementarity* and *positive externality* are satisfied. The largest Nash equilibrium is (A_1, B_1) and the smallest is (A_2, B_2). The largest Nash equilibrium is coalition-proof in both $(G, \overline{\Sigma})$ and $(G, \underline{\Sigma})$. The smallest Nash equilibrium is coalition-proof only in $(G, \underline{\Sigma})$.

Table. 1 Example 1

1 \ 2	B_1	B_2
A_1	3, 3	0, 2
A_2	2, 0	1, 1

As shown in Example 1, some Nash equilibrium is always coalition-proof for each CCS. However, not all Nash equilibria are coalition-proof. This is the difference between the coalition-proof Nash equilibrium in the game with *strategic complementarity* and that in the game with *strategic substitution*.

3.3 Weakly Dominated Strategy and Coalition-proof Nash Equilibrium

In this subsection, we assume that X_i is a finite set for each $i \in N$.[6]

[4] Let $G = [N, X, (u_i)_{i \in N}]$ be a game. Game G satisfies the *strategic complementarity* if $G' = [N, X, (-u_i)_{i \in N}]$ satisfies *strategic substitution*.

[5] A serially undominated set is the set of pure strategies that survive the iterative elimination of strictly dominated strategies.

[6] We briefly mention the case of infinite strategy sets later.

Definition 6 Let $Y_j \subseteq X_j$ for each $j \in N$. A strategy for $i \in N$, $y_i \in Y_i$, is a *weakly dominated* strategy in $\prod_{j \in N} Y_j$ if there is $z_i \in Y_i$ such that $u_i(z_i, y_{-i}) \geq u_i(y_i, y_{-i})$ for each $y_{-i} \in \prod_{j \neq i} Y_j$ and $u_i(z_i, y_{-i}) > u_i(y_i, y_{-i})$ for at least one $y_{-i} \in \prod_{j \neq i} Y_j$. A strategy y_i is an *undominated* strategy for i in $\prod_{j \in N} Y_j$ if y_i is not weakly dominated by any other strategy in Y_i.

We use the weak dominance relation based on pure strategies.[7] Note that the weak dominance relation is *transitive*[8] and *asymmetric*.[9]

Definition 7 (Iterated elimination of weakly dominated strategies) Let $X^0 \equiv \prod_{j \in N} X_j$. For each $i \in N$ and each $m \in \mathbb{Z}_{++}$, let X_i^m denote the set of strategies for i such that any $x_i \in X_i^{m-1} \backslash X_i^m$ is a weakly dominated strategy in $X^{m-1} \equiv \prod_{j \in N} X_j^{m-1}$. Suppose that at least one weakly dominated strategy is eliminated if weakly dominated strategies exist at any round of elimination. Let X^∞ be the set of strategy profiles such that no further strategy can be eliminated for each player.

For each game $G = [N, (X_i)_{i \in N}, (u_i)_{i \in N}]$, let $G^m = [N, (X_i^m)_{i \in N}, (u_i^m)_{i \in N}]$ ($m \in \mathbb{Z}_+ \cup \{\infty\}$) denote a game in which the set of strategy profiles is X^m and $u_i^m(x) = u_i(x)$ for each $i \in N$ and each $x \in X^m$. Hence, G^0 is the original game G and G^∞ is the game in which further elimination of weakly dominated strategies is not possible.

Remark 2 A coalition-proof Nash equilibrium may consist of weakly dominated strategies; therefore, a coalition-proof Nash equilibrium may be eliminated through the iterative elimination of weak dominated strategies. Peleg (1997) presents the example that is shown in Table 2. Profile (A_1, B_1) is the unique coalition-proof Nash equilibrium and A_1 and B_1 are dominated strategies if $c_i > b_i$ and $a_i > c_i$ for each $i \in \{1, 2\}$. This equilibrium cannot survive the iterative elimination of weakly dominated strategies.

Transitivity and asymmetry, together with finiteness of strategy spaces, imply that any dominated strategy is dominated by an undominated strategy and any Nash equilibrium of G^{m+1} is also a Nash equilibrium of G^m for each $m \in \mathbb{Z}_+$.

[7] Börgers (1993) provides an interesting justification for pure-strategy weak dominance.

[8] For each $i \in N$ and for each triplet $x_i, y_i, z_i \in Y_i \subseteq X_i$, if x_i weakly dominates y_i on $\prod_{j \in N} Y_j$ and y_i weakly dominates z_i on $\prod_{j \in N} Y_j$, then x_i weakly dominates z_i on $\prod_{j \in N} Y_j$

[9] For each pair $x_i, y_i \in Y_i \subseteq X_i$, if x_i weakly dominates y_i on $\prod_{j \in N} Y_j$, then y_i *does not* weakly dominate x_i

Table. 2 Coalition-proof Nash equilibrium consists of weakly dominated strategies

1 \ 2	B_1	B_2
A_1	a_1, a_2	b_1, a_2
A_2	a_1, b_2	c_1, c_2

Lemma 2 For each $m \in \mathbb{Z}_+$, $NE(G^{m+1}) \subseteq NE(G^m)$.

Proof. Suppose, to the contrary, that there is $x \in NE(G^{m+1}) \backslash NE(G^m)$ for some $m \in \mathbb{Z}_+$. There exist $i \in N$ and $y_i \in X_i^m \backslash X_i^{m+1}$ such that $u_i(y_i, x_{-i}) > u_i(x)$. Let $x_i' \in \arg \max_{z_i \in X_i^m \backslash X_i^{m+1}} u_i(z_i, x_{-i}) - u_i(x_i, x_{-i})$. Since $X_i^m \backslash X_i^{m+1}$ is finite, x_i' is well-defined. Since $x_i' \in X_i^m \backslash X_i^{m+1}$, there is $x_i'' \in X_i^m$ that weakly dominates x_i'. By the definition of x_i', $u_i(x_i'', x_{-i}) = u_i(x_i', x_{-i})$. Since $x \in NE(G^{m+1})$, $x_i'' \notin X_i^{m+1}$. Thus, x_i'' is weakly dominated by some $x_i''' \in X_i^m$ and $u_i(x_i''', x_{-i}) \geq u_i(x_i'', x_{-i}) > u_i(x)$. By the definition of Nash equilibria, $x_i''' \in X_i^m \backslash X_i^{m+1}$. Along this way, since $X_i^m \backslash X_i^{m+1}$ is finite, there is $\bar{x}_i \in X_i^m \backslash X_i^{m+1}$ that is not weakly dominated by any other strategy in $X_i^m \backslash X_i^{m+1}$. However, $\bar{x}_i$ is a weakly dominated strategy in X^m. Thus, there is $\widehat{x}_i \in X_i^{m+1}$ which weakly dominates $\widehat{x}_i$ and $u_i(x) < u_i(\bar{x}_i, x_{-i}) \leq u_i(\widehat{x}_i, x_{-i})$, which implies that $x \notin NE(G^{m+1})$. ■

Proposition 3 shows that the set of coalition-proof Nash equilibria shrinks or does not change as the round of elimination proceeds.

Proposition 3 Suppose that X_i is finite for each $i \in N$ and that an aggregative game G satisfies *monotone externality* and *strategic substitution*. Then, $CPNE(G^{m+1}) \subseteq CPNE(G^m)$ for each $m \in \mathbb{Z}_+$.

Proof. Let $m \in \mathbb{Z}_+$. Note that G^m and G^{m+1} satisfy *monotone externality* and *strategic substitution*. By Lemma 2, $NE(G^{m+1}) \subseteq NE(G^m)$. By Proposition 1, $NE(G^{m+1}) = CPNE(G^{m+1})$ and $NE(G^m) = CPNE(G^m)$. ■

Monotone externality and *strategic substitution* play an important role in Proposition 3. As the following examples indicate, if one of these conditions fails, then $CPNE(G^{m+1}) \backslash CPNE(G^m) \neq \emptyset$ for some m.

Example 2 Consider the game in Table 3, which corresponds to the case of $a_1 = a_2 = 2$, $b_1 = b_2 = 0$, and $c_1 = c_2 = 1$ in Table 2. Let A_k and B_k $(k = 1, 2)$ be such that $A_k, B_k \in \mathbb{R}$, $A_1 > A_2$, and $B_1 > B_2$. In this case, *positive externality* is satisfied, but *strategic substitution* is not. Clearly, A_1 and B_1 are dominated strategies. After these strategies are eliminated, (A_2, B_2) is the unique coalition-proof Nash equilibrium, but it is not coalition-proof in the original game.

Table. 3 Example 2

1 \ 2	B_1	B_2
A_1	2, 2	0, 2
A_2	2, 0	1, 1

Example 3 Consider the game in Table 4, in which $A_1 < A_2 < A_3$ and $B_1 < B_2 < B_3$.

Table. 4 Example 3

1 \ 2	B_1	B_2	B_3
A_1	0, 40	40, 40	40, 40
A_2	10, 41	45, 40	40, 35
A_3	20, 38	50, 30	40, 20

Because

$$
\begin{aligned}
&u_1(A_1, B_3) - u_1(A_3, B_3) = 0 > u_1(A_1, B_2) - u_1(A_3, B_2) = -10 \\
&> u_1(A_1, B_1) - u_1(A_3, B_1) = -20, \\
&u_1(A_1, B_3) - u_1(A_2, B_3) = 0 > u_1(A_1, B_2) - u_1(A_2, B_2) = -5 \\
&> u_1(A_1, B_1) - u_1(A_3, B_1) = -10, \text{ and} \\
&u_1(A_2, B_3) - u_1(A_3, B_3) = 0 > u_1(A_2, B_2) - u_1(A_3, B_2) = -5 \\
&> u_1(A_2, B_1) - u_1(A_3, B_1) = -10,
\end{aligned}
$$

and

$$u_2(A_3, B_1) - u_2(A_3, B_2) = 8 > u_2(A_2, B_1) - u_2(A_2, B_2) = 1$$
$$> u_2(A_1, B_3) - u_2(A_1, B_3) = 0,$$
$$u_2(A_3, B_1) - u_2(A_3, B_3) = 18 > u_2(A_2, B_1) - u_2(A_2, B_3) = 6$$
$$> u_2(A_1, B_1) - u_2(A_2, B_3) = 0, \text{ and}$$
$$u_2(A_3, B_2) - u_2(A_3, B_3) = 10 > u_2(A_2, B_2) - u_2(A_2, B_3) = 5$$
$$> u_2(A_1, B_2) - u_2(A_1, B_3) = 0,$$

strategic substitution is satisfied. However, since $u_1(A_2, B_1) < u_1(A_2, B_2)$ and $u_1(A_2, B_1) < u_1(A_2, B_2)$, *monotone externality* is not satisfied. Profile (A_1, B_3) is the only coalition-proof Nash equilibrium, but it consists of weakly dominated strategies. Strategies A_2 and B_2 are also weakly dominated strategies. After the elimination of A_2, A_3, B_2, and B_3, the only surviving strategy profile is (A_3, B_1), which is trivially a coalition-proof Nash equilibrium. However, it is not coalition-proof in the original game.

A game is *dominance solvable* in the sense of weak domination if X^∞ consists of only one element.

Corollary 2 Suppose that an aggregative game G satisfies *monotone externality* and *strategic substitution* and that X_i is finite for each $i \in N$. (a) Whenever G^∞ has a pure-strategy Nash equilibrium, G has a coalition-proof Nash equilibrium that is not eliminated by iterative weak dominance. (b) If G is dominance solvable, the unique surviving strategy profile is a coalition-proof Nash equilibrium in G. (c) Nash equilibria of G^∞ do not Pareto-dominate each other.

Proof. By Propositions 1 and 3, (a) is satisfied. The direct application of (a) leads to (b). By Proposition 1 and the definition of coalition-proof Nash equilibria, (c) holds. ■

Corollary 2 shows that the relationship between coalition-proofness and the iterative elimination of weakly dominated strategies is different from that between coalition-proofness and the iterative elimination of strictly dominated strategies. Moreno and Wooders (1996) examine the relationship between the coalition-proof Nash equilibrium and the iterative elimination of strictly dominated strategies. They treat a game with finite strategy sets and show that if there exists a profile of serially undominated strategies that Pareto-dominates all the other serially undominated strategies, then it is a coalition-proof Nash equilibrium. Milgrom and Roberts (1996) extend their result to a game with infinite strategy spaces under the condition of *strategic complementarity*. These two papers also show that a profile of serially undominated strategies is the only coalition-proof Nash equilibrium in a dominance solvable game.

However, when we adopt weakly domination, serially undominated strategy profiles are not necessarily coalition-proof. The difference between iterative weak domination and iterative strict domination is prominent in dominance solvable games. While the unique profile of serially undominated strategies is a coalition-proof Nash equilibrium when iterative strict dominance is analyzed, this is not necessarily true when iterative weak dominance is considered. In Example 2, (A_2, B_2) is the unique strategy profile that consists of serially undominated strategies in the sense of weak domination, but it is not coalition-proof. This also applies to Example 3. In general, Pareto-superior serially undominated Nash equilibrium is not coalition-proof if an aggregative game fails to satisfy *monotone externality* or *strategic substitution*. Corollary 2 presents a sufficient condition for an aggregative game under which some serially undominated strategy profile in the sense of weak domination is coalition-proof, analogous to the iterative elimination of strictly dominated strategies.

Finally, we make several remarks. The first is related to the assumption of finite strategy spaces. This assumption is used only in the proof of Lemma 2. The assumption guarantees that any weakly dominated strategy is weakly dominated by an **undominated** strategy. By this property, any Nash equilibrium of G^{m+1} is also a Nash equilibrium of G^m for each m. However, when the strategy sets are infinite, a strategy may be weakly dominated by another strategy that is weakly dominated by another weakly dominated strategy, and so on. Due to such an infinite sequence of dominance relations, a Nash equilibrium of G^{m+1} may not be that of G^m for some m; hence, our main results may not hold in the case of infinite strategy spaces. However, results similar to our earlier results can be obtained even in the case of infinite strategy spaces if only the strategy that is weakly dominated by an **undominated strategy** is eliminated. Under this iterative dominance concept, since any eliminated strategy is weakly dominated by an undominated strategy, $NE(G^{m+1}) \subseteq NE(G^m)$ for each m.

Second, even if the aggregative game satisfies *monotone externality* and *strategic substitution*, not all coalition-proof Nash equilibria survive the iterative elimination of weakly dominated strategies. The following example illustrates this.

Example 4 Consider the game in Table 5, in which $A_1 > A_2$ and $B_1 > B_2$. Since $u_1(x_1, B_1) < u_1(x_1, B_2)$ for each $x_1 \in \{A_1, A_2\}$ and $u_2(A_1, x_2) < u_2(A_2, x_2)$ for each $x_2 \in \{B_1, B_2\}$, *negative externality* holds. Since $u_1(A_2, B_1) - u_1(A_1, B_1) > u_1(A_2, B_2) - u_1(A_1, B_2) = 0$ and $u_2(A_2, B_2) < u_2(A_2, B_1)$, *strategic substitution* also holds. Profiles (A_2, B_1) and (A_1, B_2) are coalition-proof. However, A_1 and B_2 are both weakly dominated strategies. Hence, while (A_1, B_2) is eliminated, (A_2, B_1) survives.

Table. 5 Example 4

1 \ 2	B_1	B_2
A_1	$-3,\ 0$	$0,\ 0$
A_2	$-2,\ 2$	$0,\ 1$

Third, since the iterative elimination of weakly dominated strategies is an order-dependent procedure, it may seem insignificant to examine the relationship between coalition-proofness and the elimination of weakly dominated strategies. However, our results can be applied to an order-independent procedure which is proposed by Marx and Swinkels (1997) and Marx (1999), called the *elimination of nicely weakly dominated strategies*. For each $i \in N$ and each pair $x_i,\ x_i' \in X_i$, x_i *nicely weakly dominates* x_i' *on* X if x_i weakly dominates x_i' on X and for each $x_{-i} \in X_{-i}$, if $u_i(x_i, x_{-i}) = u_i(x_i', x_{-i})$, then $u_j(x_i, x_{-i}) = u_j(x_i', x_{-i})$ for each $j \in N$. The concept of nice weak dominance is also defined for $Y \subseteq X$. Eliminating the nicely weakly dominated strategies, we can construct a sequence of games $\{G^m\}_{m=0}^{\infty}$ as in the standard iterative weak domination. Note that if x_i nicely weakly dominates x_i', then x_i weakly dominates x_i'. Thus, together with the finiteness of strategy sets, $NE(G^{m+1}) \subseteq NE(G^m)$ for each $m \in \mathbb{Z}_+$. By using Lemma 1, we can show that $CPNE(G^{m+1}) \subseteq CPNE(G^m)$ for each m under *monotone externality* and *strategic substitution*.

4 Concluding Remarks

We investigate a coalition-proof Nash equilibrium of an aggregative game with *monotone externality* and *strategic substitution*. Due to the recursive nature of the equilibrium, it is difficult to examine which properties are satisfied by the coalition-proof Nash equilibrium even in a restricted class of games such as aggregative games.

We show that in the aggregative game with *monotone externality* and *strategic substitution*, the set of coalition-proof Nash equilibria coincide with the set of Nash equilibria. The equivalence of Nash equilibria and coalition-proof Nash equilibria implies the results of Yi (1999) and Shinohara (2005). We can conclude that this equivalence is the fundamental mechanism for the interesting properties of the coalition-proof Nash equilibrium reported by Yi (1999) and Shinohara (2005).

In the aggregative game, the coalition-proof Nash equilibrium is shown to have other interesting properties. We show that (a) the coalition-proof Nash equilibrium always assigns the same strategy under any admissible CCS and (b) some coalition-proof Nash equilibria, but not all, survive the iterative elimination of weak dominated strategies. Outside our class of games, (a) and (b) are not necessarily observed. The equivalence between Nash equilibria and coalition-proof Nash equilibria also underlies these results. As a result of our analyses, we can conclude that the various interesting properties of coalition-proof Nash equilibrium are based on the equivalence of the two equilibria.

Since the strategy set is a subset of the real line and the payoff of any agent depends not on the composition of the others' strategies but on the sum of them in the aggregative game, our analysis may appear restrictive. Whether the properties of the coalition-proof Nash equilibrium hold or not under more general strategy spaces or without the aggregative nature is left for future work.

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