Appendix: Proofs

Proof of Lemma 1.
We obtain (2) since $\underline{b} = b_M - d/2$ by the uniform population distribution on $\mathcal{B}$ and

$$\frac{b_M}{1+\beta} - \underline{b} \gtreqless 0 \ \text{ if } \ d \gtreqless \frac{2\beta b_M}{1+\beta}. \quad \blacksquare$$

Proof of Lemma 2.
Regarding (3), we find that $a_M - db_M/(1+\beta) \geq \underline{a}$ if and only if $d \leq (1+\beta)/2b_M$ since $\underline{a} = a_M - 1/2$ by the uniform population distribution on $\mathcal{A}$. For (4), since $\underline{a} = a_M - 1/2$ and $\underline{b} = b_M - d/2$, we find that $a_M - d\underline{b} \geq \underline{a}$ if and only if $d^2 - 2b_M d + 1 \geq 0$, which is equivalent to $d \geq b_M + \sqrt{b_M^2 - 1}$ or $d \leq b_M - \sqrt{b_M^2 - 1}$. $\blacksquare$

Proof of Proposition 1.
(i) By $b_M \leq (1+\beta)/(2\sqrt{\beta})$ and (5),

$$\frac{2b_M\beta}{1+\beta} \leq \frac{1+\beta}{2b_M}.$$

In addition, if $1 \leq b_M \leq (1+\beta)/(2\sqrt{\beta})$, then

$$\frac{2b_M\beta}{1+\beta} \leq b_M - \sqrt{b_M^2 - 1}.$$

By Lemmas 1 and 2 and the last inequality, $a_R(a_M, \underline{b}) = a_M - d\underline{b}$ if $d \leq 2\beta b_M/(1+\beta)$. Using these lemmas, we illustrate the optimal representatives for the median residents in Figure 2. From this figure, we obtain $(a_R^*,\ b_R^*)$ in (6).

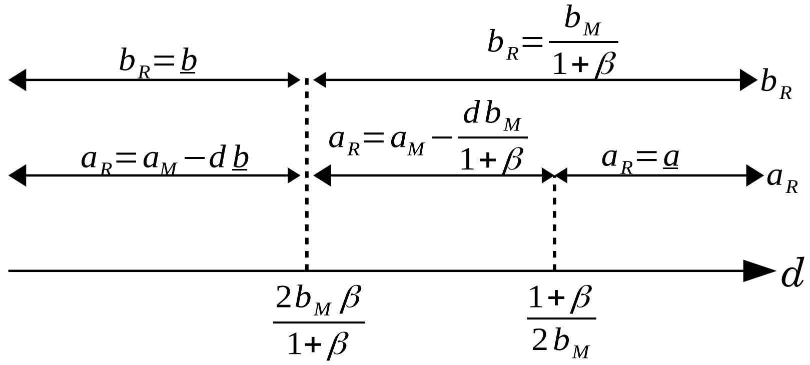

Fig. 2 Case (i)

(ii) Since $b_M > (1+\beta)/(2\sqrt{\beta})$ (≥ 1),

$$\frac{1+\beta}{2b_M} < \frac{2b_M\beta}{1+\beta} \text{ and } b_M - \sqrt{b_M^2 - 1} < \frac{2b_M\beta}{1+\beta} < b_M + \sqrt{b_M^2 - 1}. \quad (11)$$

The first condition holds by (5), and the second one can be derived by simple calculation. Using these conditions and Lemmas 1 and 2, we illustrate the optimal representatives for the median residents in Figure 3. From this figure, we obtain (a_R^*, b_R^*) in (7).

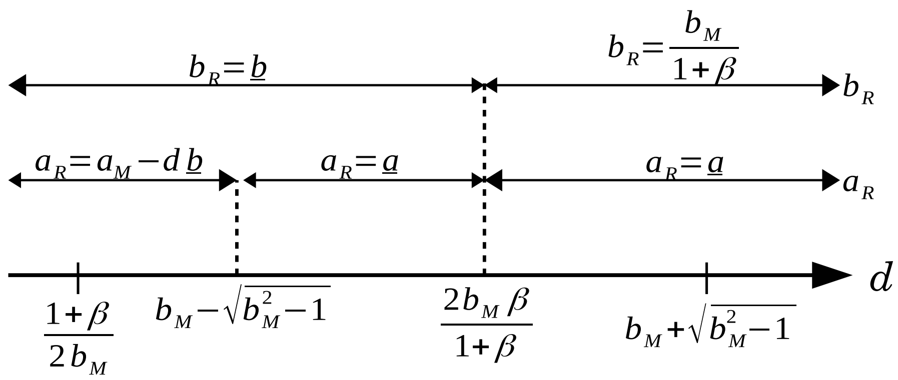

Fig. 3 Case (ii)

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Proof of Proposition 2.
Suppose that (8) or (9). By (6) and (7), $x^{NB*} = \underline{a} + db_M/(1+\beta)$. Since $\underline{a} = a_M - 1/2$,

$$x^{NB*} - x^{D*} = \underline{a} + \frac{db_M}{1+\beta} - a_M = \frac{2db_M - (1+\beta)}{2(1+\beta)} > 0 \text{ if } d > \frac{1+\beta}{2b_M}.$$

$d > (1+\beta)/(2b_M)$ holds by (8). This result also holds if (9) holds since $d > 2b_M\beta/(1+\beta)$ implies that $d > (1+\beta)/(2b_M)$ when $b_M > (1+\beta)/(2\sqrt{\beta})$ by (5).

Suppose that (10). By (7), $x^{NB*} = \underline{a} + d\underline{b}$. By $\underline{a} = a_M - 1/2$ and $\underline{b} = b_M - d/2$,

$$x^{NB*} - x^{D*} = \underline{a} + d\underline{b} - a_M = -\frac{d^2 - 2b_M d + 1}{2} > 0 \text{ if } b_M - \sqrt{b_M^2 - 1} < d < b_M + \sqrt{b_M^2 - 1}.$$

For each $d > 0$, if d satisfies $b_M - \sqrt{b_M^2 - 1} < d \leq 2b_M\beta/(1+\beta)$, then it satisfies $b_M - \sqrt{b_M^2 - 1} < d < b_M + \sqrt{b_M^2 - 1}$ because $b_M + \sqrt{b_M^2 - 1} > 2b_M\beta/(1+\beta)$ by (11). ■