Coalition-proof Nash Equilibria in a Normal-form Game and its Subgames

Ryusuke Shinohara

Faculty of Economics, Shinshu University, 3-1-1, Asahi, Matsumoto, Nagano, 390-8621, JAPAN.

ryusukes@shinshu-u.ac.jp

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Abstract

The relationship between coalition-proof (Nash) equilibria in a normal-form game and those in its subgame is examined. A subgame of a normal-form game is a game in which the strategy sets of all players in the subgame are subsets of those in the normal-form game. In this paper, focusing on a class of aggregative games, we provide a sufficient condition for the aggregative game under which every coalition-proof equilibrium in a subgame is also coalition-proof in the original normal-form game. The stringency of the sufficient condition means that a coalition-proof equilibrium in a subgame is rarely a coalition-proof equilibrium in the whole game.

Keywords Coalition-proof equilibrium; Aggregative games; No unilateral benefit; Monotone externality; Strategic substitutability.

JEL Classificaion Number C72

1 Introduction

In this paper, we consider the relationship between coalition-proof (Nash) equilibria in a normal-form game and those in its subgame. A subgame of a normal-form game is a restricted game in which the strategy sets for all players in the subgame are subsets of those in the normal-form game. Gilboa *et al.* [1990] called such a restricted game a subgame.

A Nash equilibrium of a subgame is not necessarily a Nash equilibrium of the original game. Ray [2001] studied sufficient conditions under which a Nash equilibrium of a subgame is that of the original game. He showed that, if the games satisfy the condition of *no unilateral benefit* (NUB), then a Nash equilibrium of a subgame is also a Nash equilibrium in the original normal-form game. The condition of NUB requires that no players can achieve a better payoff by playing a strategy outside the subgame, keeping the strategy of the others fixed. Ray [2001] also reported that every strong equilibrium in a subgame is also a strong equilibrium in the original game under the condition of *no coalitional benefit* (NCB), which requires that no group of players can be better off by playing strategy profiles outside the subgame.

However, surprisingly, coalition-proof equilibria of a subgame are not necessarily those in the original game even if the original game and its subgame satisfy NCB. Ray [2001] provided an example in which a two-person normal-form game and its subgame satisfy NCB, the set of coalition-proof equilibria in the subgame and that in the original game are both non-empty, and their intersection is empty. Ray [2001] conjectured that the sufficient conditions would be very strong due to the recursive nature of coalition-proofness, and he pointed out the difficulty of establishing such sufficient conditions.

In this paper, we investigate under which conditions a coalition-proof equilibrium of a subgame is also that of the original game, focusing on a class of *aggregative games*, which is a simple but well-studied class of games in economics. The aggregative game is such that strategy sets of all players are subsets of the real line and the payoffs of every player depend on his strategy and on the sum of the strategy of others. Even if our attention is limited to the aggregative games, there might be a coalition-proof equilibrium of a subgame such that the coalition-proof equilibrium of the subgame is not a coalition-proof equilibrium in the whole game. Thus, we impose not only a condition on the whole game but also a condition on a subgame. The main result in this paper is as follows: every coalition-proof equilibrium in a subgame is also coalition-proof in the original aggregative game if (i) the original game and its subgame satisfy NUB and (ii) the original aggregative game satisfies the condition of *monotone externality* and that of *strategic substitutability.* Monotone externality requires that a switch in a player's strategy change the payoffs to all the other players in the same direction. Strategic substitutability means that the incentive to a player to reduce his strategy becomes stronger as the sum of the strategy of the other players increases. The conditions of monotone externality and strategic substitutability are satisfied in many normal-form games that have interested economists, such as the standard Cournot oligopoly game and voluntary contribution games to public goods. We finally present several examples to demonstrate that each of monotone externality and strategic substitutability plays a significant role in the main result. In conclusion, some stringent conditions are imposed appropriately so that a coalition-proof equilibrium in a subgame is also a coalition-proof equilibrium in the original game. Hence, we can say that a coalition-proof equilibrium is rarely coalition-proof in the whole game.

Yi [1999] and Shinohara [2005] examined an aggregative game with monotone externality and strategic substitutability. In this game, Yi [1999] showed that a coalition-proof equilibrium is equivalent to the Pareto superior Nash equilibrium and Shinohara [2005] proved that the sets of coalition-proof equilibria based on different two dominance relations have an inclusive relation. Our results clarify a property of coalition-proof equilibria that is other than the result of the earlier literature in this class of games.

2 The Model

We consider two normal-form games, $\Gamma_1 = [N, (S_{1i})_{i\in N}, (u_{1i})_{i\in N}]$ and $\Gamma_2 = [N, (S_{2i})_{i\in N}, (u_{2i})_{i\in N}]$. For the game Γ_k $(k = 1, 2)$, $N = \{1, 2, \ldots, n\}$ is the set of players, S_{ki} is a strategy set of player $i \in N$, and $u_{ki} : \prod_{j\in N} S_{kj} \to \mathbb{R}$ is a payoff function for player i. For all coalitions $J \subseteq N$, the complement of J is denoted by $-J$. Let us denote $\prod_{i\in J} S_{ki}$ by S_{kJ}. For notational convenience, we denote $\prod_{j\in N} S_{kj}$ by S_k for all $k \in \{1, 2\}$. In this paper, we focus solely on pure strategies.

Definition 1 (Gilboa, *et al.*, 1990) A normal-form game Γ_2 is a *subgame* of Γ_1 if the strategy set of all players in Γ_2 is a subset of his strategy set in Γ_1: $S_{2i} \subseteq S_{1i}$ for all $i \in N$, and $u_{2i}(s) = u_{1i}(s)$ for all i and for all $s \in S_1 \cap S_2$.

If Γ_2 is a subgame of Γ_1, the sets of strategies for all players in Γ_2 are subsets of those in Γ_1, and payoffs to all players in Γ_2 are equal to those in Γ_1 at corresponding strategy profiles.

Definition 2 (No Unilateral Benefit (NUB)) Let Γ_2 denote a subgame of Γ_1. The games satisfy NUB if, for all $i \in N$, for all $t_{1i} \in S_{1i} \setminus S_{2i}$, and for all $s_{2\,N\setminus\{i\}} \in S_{2\,N\setminus\{i\}}$, there is $t_{2i} \in S_{2i}$ such that $u_{2i}(t_{2i}, s_{2\,N\setminus\{i\}}) \geq u_{1i}(t_{1i}, s_{2\,N\setminus\{i\}})$.

The condition of NUB requires that no players can achieve a better payoff by playing a strategy outside the subgame, keeping the strategies of others fixed.

Definition 3 (No Coalitional Benefit (NCB)) Let Γ_2 denote a subgame of Γ_1. The games satisfy NCB if, for all $J \subseteq N$, for all $t_{1J} \in S_{1J} \setminus S_{2J}$, and for all $s_{2N\setminus J} \in S_{2N\setminus J}$, there exists $t_{2J} \in S_{2J}$ such that $u_{2i}(t_{2J}, s_{2N\setminus J}) \geq u_{1i}(t_{1J}, s_{2N\setminus J})$ for all $i \in J$.

Condition NCB requires that no group of players can be better off by playing strategy profiles outside the subgame. Clearly, if Γ_1 and Γ_2 satisfy NCB, then these games satisfy NUB.

In the following, the equilibrium concepts of a normal-form game are introduced. The first is the notion of *strong equilibrium* introduced by Aumann [1959].

Definition 4 (Strong equilibrium) A strategy profile $s^* \in S_k$ is a strong equilibrium of Γ_k if there exist no coalition $J \subseteq N$ and no strategy profile $\widetilde{s}_J \in S_{kJ}$ such that $u_{ki}(\widetilde{s}_J, s^*_{-J}) > u_{ki}(s^*)$ for all $i \in J$.

A strong equilibrium is a strategy profile that is immune to all possible coalitional deviations. Thus, a strong equilibrium is a Nash equilibrium, but the converse is not necessarily true.

The second notion is a *coalition-proof equilibrium*, which was introduced by Bernheim *et al.* [1987]. Before introducing the notion of coalition-proof equilibria, we need to present a *restricted game*. For all normal-form games Γ_k, a *restricted game* with respect to a strategy profile $s \in S_k$ and a coalition $J \subseteq N$ denotes the game induced on the coalition J by strategies s_{-J}: $\Gamma_k^{J,s} = [J, (S_{ki})_{i\in J}, (u'_{ki})_{i\in J}]$, where $u'_i : \prod_{j\in J} S_{kj} \to \mathbb{R}$ is given by $u'_{ki}(t_J) = u_{ki}(t_J, s_{-J})$ for all $i \in J$ and $t_J \in \prod_{j\in J} S_{kJ}$.

Definition 5 A *coalition-proof equilibrium* $(s^*_1, \ldots, s^*_n)$ is defined inductively with respect to the number of players t:

- When $t = 1$, for all $i \in N$, s^*_i is a coalition-proof equilibrium for $\Gamma_k^{\{i\},s^*}$ if $s^*_i \in \arg\max\ u_{ki}(s_i, s^*_{-i})$ s.t. $s_i \in S_{ki}$.
- Let $T \subseteq N$ with $t = \#T \geq 2$. Assume that coalition-proof equilibria have been defined for all normal-form games with fewer players than t. Consider the restricted game Γ_k^{T,s^*} with t players.
  - A strategy profile $s^*_T \in S_{kT}$ is called *self-enforcing* if, for all $J \subsetneq T$, s^*_J is a coalition-proof equilibrium of Γ_k^{J,s^*}.
  - A strategy profile s^*_T is a coalition-proof equilibrium of Γ_k^{T,s^*} if it is a self-enforcing strategy profile and there is no other self-enforcing strategy profile $\widehat{s}_T \in S_{kT}$ such that $u_{ki}(\widehat{s}_T, s^*_{-T}) > u_{ki}(s^*_T, s^*_{-T})$ for all $i \in T$.

A coalition-proof equilibrium is clearly a Nash equilibrium in every normal-form game. Since a coalition-proof equilibrium is stable only against *self-enforcing* coalitional deviations, the set of coalition-proof equilibria contains that of strong equilibria.

Proposition 1 (Ray, 2001) Let Γ_2 denote a subgame of Γ_1. (i) Any Nash equilibrium of Γ_2 is a Nash equilibrium of Γ_1 if NUB holds. (ii) Every strong equilibrium in Γ_2 is a strong equilibrium in Γ_1 if NCB holds.

Example 1 indicates that a coalition-proof equilibrium in a subgame is not necessarily coalition-proof in the original game under the condition of NCB.

Example 1 (Ray, 2001) Consider the two-player games in Tables 1 and 2. In the two normal-form games, player 1 chooses rows, and player 2 chooses columns. A vector in each cell represents a vector of payoffs, in which the first entry is player 1's payoff and the second entry is player 2's payoff. Note that Γ_2 is a subgame of Γ_1, and Γ_1 and Γ_2 satisfy NCB. In these games, a profile of strategies (B, L) is a Nash equilibrium. However, (B, L) is coalition-proof in Γ_2, while (B, L) is not coalition-proof in Γ_1.

⟨ Insert Tables 1 and 2 here⟩

3 Results

In this section, we focus on a class of *aggregative games*. In the aggregative games, the strategy set of each player is a subset of the real line and the payoff function of every player depends on his strategy and on the aggregate strategy of all other players, which is formally defined as follows:

Definition 6 (Aggregative games) A game Γ_k is an *aggregative game* if (i) $S_{ki} \subseteq \mathbb{R}$ for all $i \in N$ and (ii), for all $i \in N$, all $s_i \in S_{ki}$, and all $s_{-i},\ \widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j \neq i} s_j = \sum_{j \neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) = u_{ki}(s_i, \widehat{s}_{-i})$.

The next condition is that of *monotone externality.* The condition states that the payoffs to every player are either non-increasing or non-decreasing with respect to the sum of strategies of the other players.

Definition 7 (Monotone externality) A game Γ_k satisfies the condition of *monotone externality* if the game satisfies either (i) or (ii).

(i) for all $i \in N$, all $s_i \in S_{ki}$, and all s_{-i} and $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j\neq i} s_j > \sum_{j\neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) \geq u_{ki}(s_i, \widehat{s}_{-i})$ holds. (*positive externality*)

(ii) for all $i \in N$, all $s_i \in S_{ki}$, and all s_{-i} and $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $\sum_{j\neq i} s_j > \sum_{j\neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) \leq u_{ki}(s_i, \widehat{s}_{-i})$ holds. (*negative externality*)

The condition of *strategic substitutability* is such that the incentive of every player to reduce his strategy becomes higher as the sum of the strategies of other players increases.

Definition 8 (Strategic substitutability) A game Γ_k satisfies the condition of *strategic substitutability* if the following condition holds: for all $i \in N$, all $s_i,\ \widehat{s}_i \in S_{ki}$, and all $s_{-i},\ \widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $s_i > \widehat{s}_i$ and $\sum_{j\neq i} s_j > \sum_{j\neq i} \widehat{s}_j$, then $u_{ki}(\widehat{s}_i, s_{-i}) - u_{ki}(s_i, s_{-i}) > u_{ki}(\widehat{s}_i, \widehat{s}_{-i}) - u_{ki}(s_i, \widehat{s}_{-i})$.

Let Γ_1 denote an aggregative game, and let Γ_2 denote a subgame of Γ_1. It is worth noting that Γ_2 satisfies these two conditions if Γ_1 does. The following proposition provides a sufficient condition under which the set of coalition-proof equilibria in Γ_2 is included in the set of coalition-proof equilibria in Γ_1.

Proposition 2 Let Γ_1 be an aggregative game, and let Γ_2 be a subgame of Γ_1. If Γ_1 and Γ_2 satisfy NUB and Γ_1 satisfies monotone externality and strategic substitutability, then every coalition-proof equilibrium in Γ_2 is also coalition-proof in Γ_1.

Proof. Let $s^* \in S_2$ denote a coalition-proof equilibrium in Γ_2. We prove that s^* is coalition-proof in Γ_1. Let us suppose, to the contrary, that s^* is not coalition-proof in Γ_1. Then, there is a coalition $J \subseteq N$ and its strategy profile $t_J \in S_{1J}$

such that t_J is coalition-proof in Γ_1^{J,s^*} and $u_{1i}(s_J^*, s_{N\setminus J}^*) < u_{1i}(t_J, s_{N\setminus J}^*)$ for all $i \in J$. Suppose that Γ_1 and Γ_2 satisfy NUB and Γ_1 satisfies positive externality and strategic substitutability.*1 Note that s^* is also a Nash equilibrium in Γ_1 by NUB. It follows from this that $\#J \geq 2$. Note also that $t_J \notin S_{2J}$.

Lemma 1 It follows that $\sum_{j\in J\setminus\{i\}} t_j > \sum_{j\in J\setminus\{i\}} s_j^*$ for all $i \in J$.

Proof of Lemma 1. Suppose, to the contrary, that $i \in J$ exists such that $\sum_{j\in J\setminus\{i\}} t_j \leq \sum_{j\in J\setminus\{i\}} s_j^*$. By the definition of Nash equilibrium, $u_{1i}(s_J^*, s_{-J}^*) \geq u_{1i}(t_i, s_{J\setminus\{i\}}^*, s_{-J}^*)$. By the condition of positive externality, $u_{1i}(t_i, t_{J\setminus\{i\}}, s_{-J}^*) \geq u_{1i}(t_i, t_{J\setminus\{i\}}, s_{-J}^*)$. Therefore, $u_{1i}(s_J^*, s_{-J}^*) \geq u_{1i}(t_i, t_{J\setminus\{i\}}, s_{-J}^*)$, which is a contradiction. **(End of Proof of Lemma 1)**

By Lemma 1, $\sum_{i\in J}\sum_{j\in J\setminus\{i\}} t_j > \sum_{i\in J}\sum_{j\in J\setminus\{i\}} s_j^*$. Thus, $\sum_{i\in J} t_i > \sum_{i\in J} s_i^*$, which implies that there is $j \in J$ such that $t_j > s_j^*$. By the definition of a Nash equilibrium, $u_{1j}(s_j^*, s_{-j}^*) - u_{1j}(t_j, s_{-j}^*) \geq 0$. From the condition of strategic substitutability, we obtain $u_{1j}(s_j^*, t_{J\setminus\{i\}}, s_{-J}^*) - u_{1j}(t_j, t_{J\setminus\{i\}}, s_{-J}^*) > u_{1j}(s_j^*, s_{-j}^*) - u_{1j}(t_j, s_{-j}^*)$. As a result, we have $u_{1j}(s_j^*, t_{J\setminus\{i\}}, s_{-J}^*) > u_{1j}(t_j, t_{J\setminus\{i\}}, s_{-J}^*)$. Therefore, t_J is not a Nash equilibrium of Γ_1^{J,s^*}. Since every coalition-proof equilibrium is a Nash equilibrium, t_J is not coalition-proof in Γ_1^{J,s^*}. This is a contradiction. Therefore, s^* is coalition-proof in Γ_1. $\square$

The following are remarks concerning the main result.

Remark 1 In a class of aggregative games, without one of the conditions of monotone externality and strategic substitutability, a coalition-proof Nash equilibrium in a subgame may not be a coalition-proof equilibrium in the original game. In Example 2, the original game and its subgame satisfy NUB, and the original game satisfies monotone externality but not strategic substitutability. In Example 3, the original game and its subgame satisfy NUB, and the original game satisfies strategic substitutability but not monotone externality. In both examples, a coalition-proof equilibrium in a

*1 We can similarly prove the statement in the case of negative externality.

subgame is not that in the original game. Therefore, these conditions play an important role in establishing the inclusive relation between coalition-proof equilibria of a game and those in its subgames in this result in the class of aggregative games.

Example 2 Consider the two-player games, Γ_1 and Γ_2, which are shown in Tables 3 and 4, respectively. We assume that $\alpha < \beta < \gamma$. Note that these two games satisfy NUB. Note also that Γ_1 satisfies the conditions of anonymity and positive externality. However, the condition of strategic substitutability does not hold. For example, if strategic substitutability holds, then $u_{11}(\alpha,\alpha) - u_{11}(\gamma,\alpha) < u_{11}(\alpha,\beta) - u_{11}(\gamma,\beta) < u_{11}(\alpha,\gamma) - u_{11}(\gamma,\gamma)$; however, we obtain $u_{11}(\alpha,\alpha) - u_{11}(\gamma,\alpha) = 20$, $u_{11}(\alpha,\beta) - u_{11}(\gamma,\beta) = 0$, and $u_{11}(\alpha,\gamma) - u_{11}(\gamma,\gamma) = 10$ in Γ_1. The only coalition-proof equilibrium is (α,α) in Γ_2, while (β,β) is the only coalition-proof equilibrium in Γ_1. Thus, a coalition-proof equilibrium in Γ_2 is not coalition-proof in Γ_1.

⟨Insert Tables 3 and 4 here⟩

Example 3 Consider the two-player games, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, which are depicted in Tables 5 and 6, respectively. We assume that $\alpha < \beta < \gamma$. Note that $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ satisfy NUB and $\widetilde{\Gamma}_1$ satisfies the strategic substitutability condition. However, $\widetilde{\Gamma}_1$ does not satisfy the condition of monotone externality. If the monotone externality condition holds, then either $u_{11}(\beta,\alpha) \leq u_{11}(\beta,\beta) \leq u_{11}(\beta,\gamma)$ or $u_{11}(\beta,\alpha) \geq u_{11}(\beta,\beta) \geq u_{11}(\beta,\gamma)$ holds; however, we have $u_{11}(\beta,\alpha) = 0$, $u_{11}(\beta,\beta) = 40$, and $u_{11}(\beta,\gamma) = 30$. Clearly, profile (γ,α), which is coalition-proof in $\widetilde{\Gamma}_2$, is not a coalition-proof equilibrium in $\widetilde{\Gamma}_1$.

⟨Insert Tables 5 and 6 here⟩

Remark 2 The condition of monotone externality and that of strategic substitutability cannot be dropped even if a normal-form game and its subgame satisfy NCB. This is demonstrated in Example 2 and Example 3. In these examples, the original game and its subgame satisfy NCB, which is stronger than NUB. Nevertheless, the set of coalition-proof equilibria in the original game and that in the subgame are disjointed.

Remark 3 As well as the strategic substitutability condition, a condition of *strategic complementarity* has also been studied in economics. The strategic complementarity condition is formally defined as follows: for all $i \in N$, all s_i, $\widehat{s}_i \in S_{ki}$, and all s_{-i}, $\widehat{s}_{-i} \in S_{k\,N\setminus\{i\}}$, if $s_i > \widehat{s}_i$ and $\sum_{j\neq i} s_j > \sum_{j\neq i} \widehat{s}_j$, then $u_{ki}(s_i, s_{-i}) - u_{ki}(\widehat{s}_i, s_{-i}) > u_{ki}(s_i, \widehat{s}_{-i}) - u_{ki}(\widehat{s}_i, \widehat{s}_{-i})$. This condition means that an incentive of a player to increase his strategy becomes stronger as the sum of the others' strategies increases. A game in Table 7 shows that a coalition-proof equilibrium of a subgame is not necessarily the one in the whole game. Notice that this game satisfies the strategic complementarity and the negative externality condition when $\alpha < \beta$ is assumed. Consider a subgame in which the sets of strategies for both players consist only of β. While (β, β) is trivially coalition-proof in this subgame, it is not coalition-proof in the whole game.

⟨Insert Table 7 here⟩

4 Conclusion

In this paper, the relationship between coalition-proof equilibria of a normal-form game and its subgame was examined. Ray [2001] showed that a Nash equilibrium in a subgame is that in the original game under the condition of NUB and a strong equilibrium of a subgame is that of the original game under NCB. However, a coalition-proof equilibrium of a subgame is not necessarily coalition-proof in the original game even if NCB holds, and Ray [2001] pointed out the difficulty of providing a sufficient condition under which the set of coalition-proof equilibria of the original game and that of its subgame are related by inclusion. In this paper, focusing on a class of aggregative games, we investigated in which games a coalition-proof equilibrium in a subgame is also the one in the whole game. We showed that the set of coalition-proof equilibria in a subgame is included in that of coalition-proof equilibria in the whole game if the subgame and the whole game satisfy NUB and the whole game satisfies monotone externality and strategic substitutability. We also provided some examples and proved that monotone externality and strategic substitutability play a significant role in establishing the inclusion relation between the two coalition-proof equilibria. From these observations, we confirm that this inclusive relation can only be observed if several stringent conditions are imposed on games, appropriately. This would hold true in not only the class of aggregative games but also the other class of games. In this sense, we can conclude that the inclusive relation between coalition-proof equilibria in a subgame and those in the whole game is unlikely to be observed.

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Table. 1 The original game Γ_1

(Ray, 2001)

1 \ 2	L	R	Y
X	0, 0	0, 0	3, 3
T	0, 5	4, 4	0, 0
B	2, 2	5, 0	0, 0

Table. 2 A subgame Γ_2 of Γ_1

(Ray, 2001)

1 \ 2	L	R
T	0, 5	4, 4
B	2, 2	5, 0

Table. 3 A two-player game Γ_1

1 \ 2	α	β	γ
α	20, 20	20, 20	50, 0
β	20, 20	30, 30	35, 20
γ	0, 50	20, 35	40, 40

Table. 4 A subgame Γ_2 of Γ_1

1 \ 2	α	γ
α	20, 20	50, 0
γ	0, 50	40, 40

Table. 5 A two-player game $\widetilde{\Gamma}_1$

1 \ 2	α	β	γ
α	-30, 30	30, 30	40, 40
β	0, 50	40, 40	30, 30
γ	20, 20	50, 0	30, -30

Table. 6 A subgame $\widetilde{\Gamma}_2$ of $\widetilde{\Gamma}_1$

1 \ 2	α	β
β	0, 50	40, 40
γ	20, 20	50, 0

Table. 7 An example in Remark 3

2 / 1	α	β
α	2, 2	0, 2
β	2, 0	1, 1