The Possibility of Efficient Provision of a Public Good in Voluntary Participation Games[*]

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August 1, 2008

(forthcomming in *Social Choice and Welfare*)

[*] I would like to thank Koichi Tadenuma, Yukihiro Nishimura, Toshiyuki Fujita, John Weymark, Tatsuyoshi Saijo, Takehiko Yamato, Toshiji Miyakawa, Motohiro Sato, Takashi Shimizu, Nobue Suzuki, and Dirk T. G. Rübbelke for helpful comments and suggestions. I also thank the anonymous referees for detailed comments and suggestions. This research was supported by the Japanese Economic Research Foundation and the Grand-in-Aid for Young Scientists (Start-up) from the Japan Society for the Promotion of Science. Any remaining errors are my own.

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Abstract

In this study, we provide the conditions for efficient provision of a public good in a participation game in which a non-negative integer number of units of the public good can be provided. In the case in which at most one unit of the public good can be provided, we provide refinements of Nash equilibria at which agents choose only a Nash equilibrium with an efficient allocation and provide sufficient conditions for cost-sharing rules that guarantee the existence of a Nash equilibrium with an efficient allocation. In the case of a multi-unit public good, we provide a necessary and sufficient condition for the existence of a Nash equilibrium with an efficient allocation and prove that Nash equilibria are less likely to support efficient allocations if the participation of many agents is needed for efficient provision of the public good in the case of identical agents.

Keywords: Participation game, Discrete public good, Public good provision.
JEL Classification Numbers: C72, D71, H41.

1 Introduction

This paper considers a participation problem in a mechanism for the provision of a pure public good. From the theory of implementation, the construction of a mechanism can solve the "free-rider" problem in economies with public goods. For example, Bagnoli and Lipman (1989), Jackson and Moulin (1992), and Bag (1997) constructed mechanisms to implement desirable allocation rules in an economy with a discrete public good.

However, implementation theory assumes that all agents participate, and each agent lacks the right to determine whether or not to participate in the mechanism. Palfrey and Rosenthal (1984), Saijo and Yamato (1999), and Dixit and Olson (2000) pointed out the importance of the strategic behavior of agents as they decide whether or not to participate in the mechanism. In the real world, for example, in participation problems for international environmental treaties, agents often have the right to make such decisions, and they may have an incentive not to enter the mechanism, hoping that other agents will participate in the mechanism and provide a public good. This behavior generates another kind of a free-rider problem.

These authors formulated a participation game for a public good mechanism. In the game, each agent simultaneously chooses whether or not to participate in the mechanism. If an agent chooses to participate, he pays the fee requested by the mechanism, and the public good is produced. If an agent chooses not to participate, he can enjoy the public good at no cost. Palfrey and Rosenthal (1984) and Dixit and Olson (2000) analyzed the participation game for the case in which a public good can be produced in integer units and, at most, one unit of public good is produced, which is called a *participation game with a public project* hereafter. Saijo and Yamato (1999) examined a participation game with a perfectly divisible public good.

In this paper, we examine the conditions for efficient provision of a public good in a participation game when a non-negative integer number of units of the public good can be provided. First, in the participation game with a public project, we examine which types of agents' behavior achieve allocative efficiency in non-cooperative environments. Although the standard solution for non-cooperative games is the Nash equilibrium, it is well known that the participation game with a public project has many Nash equilibria and both efficient allocations and inefficient allocations are supported as Nash equilibria. Palfrey and Rosenthal (1984) pointed out that efficient provision, underprovision, and overprovision of a public good are possible at pure-strategy and mixed-strategy equilibria and that early experimental results reporting overprovision of a public good may be consistent with self-interested Nash behavior. Dixit and Olson (2000) focused on mixed-strategy Nash equilibria and argued that underprovision of a public good at the mixed-strategy equilibria undermines the Coase Theorem. However, earlier studies have not investigated which types of Nash equilibria achieve efficient allocations. In this paper, we provide refinements of Nash equilibria that support efficient allocations and investigate which behavioral assumptions ensure efficient provision of the public good, in addition to the Nash behavioral principle.

Second, in a participation game with a public project, we investigate which cost-sharing rules guarantee the existence of Nash equilibria that support efficient allocations. Palfrey and Rosenthal (1984) and Dixit and Olson (2000) assumed that participants share the cost of the project equally. Under this equal cost-sharing rule, if agents' preferences are identical, then the participation game has a Nash equilibrium with an efficient allocation. However, under this rule, when agents' preferences are heterogeneous, a Nash equilibrium with an efficient allocation does not necessarily exist. On the other hand, as is shown in this paper, an equilibrium with an efficient allocation always exists if the cost is shared among participants in proportion to their benefits from the project. We identify a class of such cost-sharing rules by providing a sufficient condition for the cost-sharing rules that guarantees the existence of a Nash equilibrium with an efficient allocation.

Finally, we extend our analysis to a participation game in which up to two units of the public good can be produced, and we provide a necessary and sufficient condition for Nash equilibria to achieve allocative efficiency. While Palfrey and Rosenthal (1984) and Dixit and Olson (2000) considered participation games with a public project and Saijo and Yamato (1999) considered a game with a perfectly divisible public good, the participation game with a discrete and multi-unit public good has not been studied. In this paper, we consider a participation game in which at most two units of the public good can be produced, which is close to a participation game with a public project. We conjecture that there is a high possibility that this participation game has an efficient Nash equilibrium since the participation game with a public project has a Nash equilibrium with an efficient allocation under some cost-sharing rule, as stated below. However, this conjecture is not true. We investigate a necessary and sufficient condition for efficient provision of a public good at Nash equilibria, focusing on the relationship between the minimal number of participants for efficient provision of a public good and the marginal benefits of agents from the public good. We discuss the possibility of efficient provision of a public good in the two-unit case.

Our results are as follows: we first examine strict Nash equilibria and strong equilibria of the participation game with a public project when the cost of the project is distributed according to the proportional cost-sharing rule. The strict Nash equilibria and strong equilibria are refinements of Nash equilibria. We show that the set of strict Nash equilibria and the set of Nash equilibria with an efficient allocation coincide under some conditions and the set of strong equilibria is included in the set of strict Nash equilibria. Hence, if agents choose a strict Nash equilibrium or a strong equilibrium, an efficient allocation is attained. This is the first contribution of this paper. Second, we prove that, if the cost of the project is distributed according to a cost-sharing rule that satisfies the conditions of budget balance, individual rationality, and positive cost-shares, then the participation game has a Nash equilibrium with an efficient allocation. Since the proportional cost-sharing rule satisfies these three conditions, we can conclude that the proportional cost-sharing rule is one of the "favorable" rules for the existence of efficient Nash equilibria in the participation game with a public project. This is the second finding of this paper.

In a participation game in which at most two units of the public good can be produced, Nash equilibria do not necessarily support efficient allocations. We characterize the Nash-equilibrium sets of participants and examine how it is possible to attain efficient allocations at Nash equilibria in the case of identical agents. We characterize the range of ratios of marginal benefits in which efficient allocations are supported as Nash equilibria. We prove that the range of ratios of marginal benefits shrinks as the number of participants increases. From this result, efficient provision of the public good can be achieved at few preference parameters if the participation of many agents is necessary for efficiently producing the public good. Therefore, we can conclude that, if the number of agents in the economy is large and efficient provision of the public good requires the participation of a large fraction of agents, then efficient provision of the public good is less likely to be attained at a Nash equilibrium. The difficulty of providing the public good efficiently in the two-unit case has not been pointed out by any earlier studies. This result would imply the difficulties to provide a discrete and multi-unit public good efficiently in general environments. This is the third contribution to the literature.

This paper is organized as follows: in Section 2, we introduce the participation game with a public project and its equilibrium concepts. Section 3 presents the results of the participation game with a public project. In Section 4, our analysis is extended to the case of a multi-unit and discrete public good. Section 5 is the conclusion of this paper.

2 A participation game with a public project

We consider the problem of undertaking a (pure) public project and distributing its cost. Let n be the number of agents. We denote the set of agents by $N = \{1, \dots, n\}$. Let $y \in \{0, 1\}$ be the public project: $y = 1$ if the project is undertaken and $y = 0$ if not. Let $\theta_i > 0$ denote agent i's *willingness to pay* for the project or *benefit* from the project. Let $x_i \geq 0$ denote a transfer from agent i. Each agent i has a preference relation which is represented by a quasi-linear utility function $V_i(y, x_i) = \theta_i y - x_i$. The cost of the project is $c > 0$.

In this paper, we assume that there exists a mechanism that implements the proportional cost-sharing rule. We consider a two-stage game. In the first stage, each agent simultaneously decides whether to participate in the mechanism or not. In the second stage, following the rules of the mechanism, only the agents that selected participation in the first stage decide whether to implement the project. As a result, only the *proportional cost-sharing allocation* for participants' preferences is achieved.

Let P be a set of participants and let $(y^P, (x_j^P)_{j \in N})$ be the outcome of the second stage when P is the set of participants. We denote $\theta_P = \sum_{j \in P} \theta_j$ for all sets of participants P: θ_P is the sum that agents in P are willing to pay for the public project. For all subsets P of N, $\#P$ means the cardinality of set P.

Assumption 1 For every set of participants P, the allocation to participants $(y^P, (x_j^P)_{j \in P})$ satisfies the following conditions:

$$
\begin{aligned}
& y^P \in \arg\max_{y \in \{0,1\}} (\theta_P - c) y, \\
& x_i^P = \frac{\theta_i}{\theta_P} c \text{ for all } i \in P \text{ if } y^P = 1, \text{ and} \\
& x_i^P = 0 \text{ for all } i \in P \text{ if } y^P = 0
\end{aligned}
$$

From Assumption 1, the public good is produced in a way that maximizes the surplus of participants. If $y^P = 1$, then the cost of the project is distributed among the participants in proportion to the benefits that the participants receive from the project. If $y^P = 0$, the participants bear no costs.

In this paper, we are not concerned with the implementation problem of the proportional cost-sharing rule. However, there are mechanisms in which the proportional cost-sharing rule is attained in equilibrium. For example, Jackson and Moulin (1992) constructed a multi-stage mechanism which implements the proportional cost-sharing rule in subgame perfect Nash equilibria and undominated Nash equilibria. In these mechanisms, agents report an estimate of the collective benefit accruing from the project and their own benefit for the project. In the equilibria of these mechanisms, agents truthfully announce the collective benefit and their own benefits.

Assumption 2 Let $P \subseteq N$ be a set of participants. For all $i \notin P$, $x_i^P = 0$ and every non-participant can also consume y^P.

Assumption 2 expresses the non-excludability of the project. With this assumption, participants bear the costs of the project, but non-participants do not. In spite of this, non-participants can benefit from the project.

Given the outcome of the second stage, the participation-decision stage can be reduced to the following simultaneous game. In the game, each agent i simultaneously chooses either $s_i = I$ (participation) or $s_i = O$ (non-participation). Let P^s be the set of participants at action profile $s = (s_1, \ldots, s_n)$. Then, each agent i obtains utility $V_i(y^{P^s}, x_i^{P^s})$ at action profile s. In other words, if the public project is undertaken, the participants share its cost in proportion to the benefits obtained from the project. Each non-participant can free-ride on the public project. On the other hand, if the project is not carried out, then the payoffs of both participants and non-participants are zero. We call this reduced game a *participation game* and formally define it as follows.

Definition 1 (Participation game) A *participation game* is represented by $G = \left[N,\, S^n = \{I,\, O\}^n,\, (U_i)_{i \in N}\right]$, where U_i is the payoff function of i, which associates a real number $U_i(s)$ with each strategy profile $s \in S^n$: if P^s designates the set of participants at s, then $U_i(s) = V_i(y^{P^s}, x_i^{P^s})$ for all i.

The notions of equilibria of the participation game are defined as follows. The Nash equilibria of the participation game are defined as usual. First, a definition is given for a strict Nash equilibrium.

Definition 2 (Strict Nash equilibrium) A strategy profile $s^* \in S^n$ is a *strict Nash equilibrium* if, for all $i \in N$ and for all $\widehat{s}_i \in S \setminus \{s_i^*\}$, $U_i(s_i^*, s_{-i}^*) > U_i(\widehat{s}_i, s_{-i}^*)$.

We focus on the strict Nash equilibria of participation games with a public project. Since all strict Nash equilibria are pure-strategy profiles, our attention is limited to the pure-strategy profiles. The results in this paper do not change even if agents can use randomized strategies. See Remark 2 for a detailed discussion.

Before defining a strong equilibrium, some notation is presented. For all $D \subseteq N$, denote the complement of D by $-D$. For all coalitions D, $s_D \in S^{\#D}$ denotes a strategy profile for D. For all $s_N \in S^n$, denote s_N by s.

Definition 3 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a *strong equilibrium of* G if there exists no coalition $T \subseteq N$ and no strategy subprofile $\widetilde{s}_T \in S^{\#T}$ such that $\sum_{i \in T} U_i(\widetilde{s}_T, s_{-T}^*) > \sum_{i \in T} U_i(s^*)$ for all $i \in T$.

A strong equilibrium is a strategy profile at which no coalition, taking the strategies of the other agents as given, can jointly deviate in a way that increases the sum of the payoffs of its members. The strong equilibrium in Definition 3 is slightly different from that originally defined by Aumann (1959). The difference lies in the possibility of monetary transfers among agents in coalitions. Our definition allows members of coalitions to freely send monetary transfers to each other, but Aumann's definition (1959) does not. Hence, in our model, members of a coalition can coordinate their participation decisions through monetary transfers. It is noteworthy that the set of strong equilibria in a game without monetary transfers generally contains a set of strong equilibria in a game with monetary transfers. However, the converse is not necessarily true. Obviously, all strict Nash equilibria and all strong equilibria are Nash equilibria. However, the set of strict Nash equilibria and the set of strong equilibria are not always related by inclusion.

Example 1 Let $N = \{1, 2, 3\}$, $\theta_1 = \theta_2 = \theta_3 = 3/4$, and $c = 1$. The payoff matrix of this example is depicted in Table 1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria. One consists of the Nash equilibria with two participants and the other is the Nash equilibrium with no participants. Only the Nash equilibria with the participation of two agents are strict Nash equilibria and strong equilibria.

⟨Insert Table 1 here.⟩

3 Results of the participation game with a public project

3.1 Strict Nash equilibria and efficient Nash equilibria of the participation game with a public project

In this subsection, we examine the relationship between strict Nash equilibria and Nash equilibria supporting efficient allocations in the participation game with a public project.

The set of feasible allocations is defined as A:

$$A = \left\{ (y, (x_j)_{j \in N}) \mid x_j \geq 0 \text{ for all } j \in N,\ y \in \{0, 1\} \text{ and } \sum_{j \in N} x_j \geq cy \right\}.$$

Assumption 3 $\theta_N > c$.

Definition 4 An allocation $(y, (x_j)_{j \in N})$ is *Pareto efficient* if there exists no feasible allocation $(\widehat{y}, (\widehat{x}_j)_{j \in N})$ such that $V_i(\widehat{y}, \widehat{x}_i) \geq V_i(y, x_i)$ for all $i \in N$ with strict inequality for at least one $i \in N$.

In this paper, we call a Nash equilibrium that supports an efficient allocation an *efficient Nash equilibrium*. Henceforth, we assume that Assumption 3 holds. By Assumption 3, the public project is undertaken at every Pareto efficient allocation. In the next lemma, we characterize the sets of participants supported as Nash equilibria.

Lemma 1 Under Assumptions 1 and 2, the following statements are satisfied.

(1.1) Let $P \subseteq N$ be such that $\theta_P > c$. Then, P is supported as a Nash equilibrium if and only if (i) $\theta_P - \theta_i \leq c$ for all $i \in P$, and (ii) $y^{P\setminus\{i\}} = 0$ if there is $i \in P$ such that $\theta_P - \theta_i = c$.

(1.2) Let $P \subseteq N$ be such that $\theta_P < c$. Then, P is attained at a Nash equilibrium if and only if $\theta_P + \theta_i \leq c$ for all $i \notin P$.

(1.3) Let $P \subseteq N$ be such that $\theta_P = c$. Then, P is a Nash-equilibrium set of participants if and only if $y^P = 1$.

Proof. First, we show (1.1). Let P be a set of participants that satisfies $\theta_P > c$. Let $(y^P, (x_j^P)_{j\in P})$ denote the allocation when P is the set of participants.

If P is supported as a Nash equilibrium, then (i) obviously holds. If $i \in P$ exists such that $\theta_P - \theta_i = c$ and $y^{P\setminus\{i\}} = 1$, then i has an incentive to deviate from I to O. Therefore, (ii) must be satisfied. Conversely, suppose that $\theta_P - \theta_i \leq c$ for all $i \in P$ and $y^{P\setminus\{i\}} = 0$ if there exists $i \in P$ such that $\theta_P - \theta_i = c$. Then, we have

$$V_i(y^P, x_i^P) = \theta_i - \frac{\theta_i}{\theta_P}c \;>\; 0 = V_i(y^{P\setminus\{i\}}, x_i^{P\setminus\{i\}}) \text{ for all } i \in P, \text{ and}$$
$$V_i(y^P, x_i^P) = \theta_i \;>\; \theta_i - \frac{\theta_i}{\theta_P + \theta_i}c = V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}}) \text{ for all } i \notin P.$$

Hence, P is supportable as a Nash equilibrium.

Second, we prove (1.2). Let P be such that $\theta_P < c$. If $\theta_P + \theta_i \leq c$ for all $i \notin P$, then P is clearly supported as a Nash equilibrium. Conversely, suppose that P is supported as a Nash equilibrium. Then, we have $V_i(y^P, x_i^P) \geq V_i(y^{P\cup\{i\}}, x_i^{P\cup\{i\}})$ for all $i \notin P$. Since $\theta_P < c$, we obtain $V_i(y^P, x_i^P) = 0$ for all $i \notin P$. If there exists agent $j \notin P$ such that $\theta_P + \theta_j > c$, then we obtain $V_j(y^{P\cup\{j\}}, x_j^{P\cup\{j\}}) = \dfrac{\theta_j}{(\theta_P + \theta_j)}(\theta_P + \theta_j - c) > 0$. This means that agent j has an incentive to deviate from O to I. Therefore, we have $\theta_P + \theta_i \leq c$ for all $i \notin P$.

Third, we show (1.3). Let P be a set of participants that satisfies $\theta_P = c$. Suppose that P is a Nash-equilibrium set of participants and $y^P = 0$. Then, non-participant $j \notin P$ obtains the payoff $\theta_j - \dfrac{\theta_j}{\theta_P + \theta_j}c > 0$ if he chooses I, and he receives 0 if he selects O. Therefore, $y^P = 1$ must be satisfied. Conversely, if $y^P = 1$, then P with $\theta_P = c$ is obviously attained at a Nash equilibrium. ■

In the following lemma, we show that there is a Nash equilibrium at which the project is carried out in the participation game.

Lemma 2 Under Assumptions 1, 2, and 3, there exists a Nash equilibrium to undertake the public project in the participation game.

Proof. Let P be a set of participants such that:

$$P \in \arg\min_{Q \subseteq N} \theta_Q \text{ such that } \theta_Q > c. \tag{1}$$

Note that there is at least one set of participants R satisfying $\theta_R > c$ by Assumption 3. If there is some $j \in P$ such that $\theta_P - \theta_j > c$, then $\theta_P > \theta_{P\setminus\{j\}} > c$. This contradicts (1). Hence, P satisfies $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$. If $y^{P\setminus\{i\}} = 0$ for all $i \in P$ with $\theta_{P\setminus\{i\}} = c$, then P is a Nash-equilibrium set of participants at which the project is undertaken. If $j \in P$ exists such that $\theta_{P\setminus\{j\}} = c$ and $y^{P\setminus\{j\}} = 1$, then $P \setminus \{j\}$ is supported as a Nash equilibrium to carry out the project. ■

Remark 1 The set of Nash equilibria in (1.1) of Lemma 1 coincides with the set of strict Nash equilibria in the participation game. The Nash equilibria in (1.2) and (1.3) are non-strict Nash equilibria. Although the game has a Nash equilibrium to undertake the project, there is not necessarily a strict Nash equilibrium in this game. Since, in the participation game, the set of strict Nash equilibria is included in the set of Nash equilibria to undertake the project, the two equilibrium sets do not always coincide. However, we can confirm from (1.1) of Lemma 1 and Lemma 2 that the participation game has a strict Nash equilibrium if the project is not carried out when the case in which the project is undertaken and the case in which it is not are indifferent for a set of participants: $y^P = 0$ for all P with $\theta_P = c$. Moreover, with the same assumption, the set of strict Nash equilibria coincides with the set of efficient Nash equilibria.

Assumption 4 For all $P \subseteq N$, if $\theta_P = c$, then $y^P = 0$.

Proposition 1 Suppose that Assumptions 1, 2, 3, and 4 are satisfied. Then, in the participation game, a strategy profile is a strict Nash equilibrium if and only if it is an efficient Nash equilibrium.

Proof. Under the assumption that $y^P = 0$ for all P with $\theta_P = c$, the participation game has a strict Nash equilibrium. If a profile of strategies is a strict Nash equilibrium, then it is a Nash equilibrium with $\theta_P > c$, and, from Lemma 1, only the strict Nash equilibria are Nash equilibria with $\theta_P > c$. ■

From Proposition 1, we confirm that the strict Nash equilibrium is a refinement of the Nash equilibrium concept that guarantees that Nash equilibria are efficient and that every efficient Nash equilibrium is realized as a strict equilibrium in the participation game under the assumption that $y^P = 0$ for all P with $\theta_P = c$. Hence, with this assumption, agents choose strict Nash equilibria if and only if allocative efficiency is achieved in equilibrium

3.2 Strong equilibria of the participation game with a public project

In this subsection, we show that efficient strong equilibria also exist, as with strict Nash equilibria, when Assumption 4 is satisfied.

3.2.1 A characterization of strong equilibria

Lemma 3 Suppose that Assumptions 1, 2, and 4 hold. Let $s^* \in S^n$ denote a strict Nash equilibrium of the participation game and let P^* be the set of participants at s^*. A strict Nash equilibrium s^* is a strong equilibrium of G if and only if there is no coalition T and no strategy subprofile $\widehat{s}_T \in S^{\#T}$ such that

$$T_I^* \subsetneq P^*,\ \theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0, \text{and } \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c, \tag{2}$$

where $T_I^* = \{i \in T | s_i^* = I\}$ and $\widehat{T}_I = \{i \in T | \widehat{s}_i = I\}$.

As a preparation for the proof of Lemma 3, we first show the following lemma.

Lemma 4 Under Assumptions 1, 2, and 4, only the coalitional deviations from a strict Nash equilibrium that satisfy (2) increase the sum of the payoffs to the members of the coalition in the participation game.

Proof. Let s^* denote a strict Nash equilibrium of the participation game. Denote the set of participants at s^* by P^*. Let T denote a coalition and let $\widehat{s}_T$ denote a profile of strategies for T. Let us denote the set of participants at $(\widehat{s}_T, s^*_{-T})$ by $\widehat{P}$. If we let $T^*_I = P^* \cap T$ and $\widehat{T}_I = \widehat{P} \cap T$, then $\widehat{P} = (P^* \setminus (T^*_I \setminus \widehat{T}_I)) \cup (\widehat{T}_I \setminus T^*_I)$. Note that $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T^*_I \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T^*_I}$.

Claim 1 If $\theta_{\widehat{P}} \geq \theta_{P^*}$, then deviations by T from s^* are not profitable: $\sum_{i \in T} U_i(s^*_T, s^*_{-T}) \geq \sum_{i \in T} U_i(\widehat{s}_T, s^*_{-T})$.

Proof of Claim 1. The sum of the payoffs of agents in T at s^* is

$$\theta_T - \frac{\theta_{T^*_I}}{\theta_{P^*}} c > 0, \tag{3}$$

and the sum at $(\widehat{s}_T, s^*_{-T})$ is

$$\theta_T - \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}} c. \tag{4}$$

Subtracting (4) from (3) yields

$$\frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*} \left(\theta_{\widehat{T}_I} - \theta_{T^*_I} \right) - \theta_{T^*_I} \left(\theta_{\widehat{T}_I \setminus T^*_I} - \theta_{T^*_I \setminus \widehat{T}_I} \right) \right).$$

Since $\theta_{\widehat{T}_I} - \theta_{T^*_I} = \theta_{\widehat{T}_I \setminus T^*_I} - \theta_{T^*_I \setminus \widehat{T}_I}$, we obtain

$$\frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*} - \theta_{T^*_I} \right) \left(\theta_{\widehat{T}_I \setminus T^*_I} - \theta_{T^*_I \setminus \widehat{T}_I} \right). \tag{5}$$

We have $\theta_{P^*} - \theta_{T^*_I} \geq 0$ because $T^*_I \subseteq P^*$. Since $\theta_{\widehat{P}} \geq \theta_{P^*}$, we obtain $\theta_{\widehat{T}_I \setminus T^*_I} \geq \theta_{T^*_I \setminus \widehat{T}_I}$. Therefore, (5) is greater than or equal to zero. **(End of the Proof of Claim 1)**

By Claim 1, the deviations by T satisfy $\theta_{P^*} > \theta_{\widehat{P}}$ if the deviations are profitable. Since $\theta_{P^*} > \theta_{\widehat{P}}$, we obtain $\theta_{T^*_I \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T^*_I}$.

Claim 2 If $\theta_{\widehat{P}} \leq c$, then the deviations by T are not profitable.

Proof of Claim 2. If $\theta_{\widehat{P}} \leq c$, the project is not undertaken, and the sum of the payoffs that the members of T receive after the deviation is zero. Clearly, the deviations by T are not profitable. **(End of the Proof of Claim 2)**

From Claim 1 and Claim 2, $\theta_{P^*} > \theta_{\widehat{P}} > c$ must be satisfied so that the deviations by T are profitable. By Lemma 1, $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$. Therefore, $\theta_{P^*} - \theta_{T^*_I \setminus \widehat{T}_I} \leq c$.

By Claim 2, $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c$. Thus, we have $\theta_{\widehat{T}_I \setminus T_I^*} > 0$. Accordingly, it follows that $\theta_{P^*} > \theta_{\widehat{P}} > c$ and $\theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0$.

Claim 3 If $T_I^* = P^*$, then the deviations by T are not profitable.

Proof of Claim 3. Note that the difference between the sum of the payoffs that the members of T receive at s^* and that at $(\widehat{s}_T, s^*_{-T})$ is equal to (5). Therefore, the two payoffs are equal if $T_I^* = P^*$. **(End of the Proof of Claim 3)**

By Claims 1, 2, and 3, only the deviations by T that satisfy (2) are profitable. ■

Proof of Lemma 3. The sufficiency of the statement is immediate from Lemma 4. Necessity is trivial. ■

Lemma 3 says that a deviation from a strict Nash equilibrium results in an improvement for the deviator if and only if the following situation exists: at a strict Nash equilibrium, some participants and non-participants form a coalition and can coordinate in such a way that both the sum of the benefits from the project of the participants decreases and the project is undertaken. In this situation, members of the coalition that change their strategies from I to O get benefits, whereas those who change from O to I suffer losses. However, by transferring part of the benefits to the latter agents, the members switching from I to O can make up for the losses. As a result, all members of the coalition can improve their payoffs after this deviation.

In contrast to the case in which monetary transfers are allowed, when monetary transfers are impossible, every strict Nash equilibrium is a strong equilibrium in the participation game. Consider deviations from a strict Nash equilibrium in which the project is undertaken. It is clear that deviations only by participants from the equilibrium and deviations only by non-participants from the equilibrium are not profitable. Thus, both participants and non-participants must deviate jointly for the deviations to be profitable. Note that at least one of the non-participants changes his strategy O to I if the payoffs to the participants increase, and, at the equilibrium, the non-participants receive the greatest payoff that can be attained in the participation game. Thus, members of the coalition that change their strategies from O to I are worse off when transfers are not allowed. Therefore, coalitions cannot deviate from a strict Nash equilibrium in a way that increases the payoffs to participants without decreasing the payoffs to non-participants. Using the same logic, Shinohara (2007) showed that the set of strict Nash equilibria and the set of strong equilibria coincide when monetary transfers are not allowed in the participation game in a mechanism that implements a class of allocation rules including the proportional cost-sharing rule.

3.2.2 Existence of a strong equilibrium

Proposition 2 Under Assumptions 1, 2, 3, and 4, a strong equilibrium exists in the participation game with a public project.

Proof. Let P^{min} be such that $P^{min} \in \arg\min_{P \subseteq N} \theta_P$ subject to $\theta_P > c$. Since P^{min} satisfies $\theta_{P^{min}} - \theta_i \leq c$ for every $i \in P^{min}$, P^{min} is supportable as a strict Nash equilibrium. Let $s^{min} \in S^n$ be the strict Nash equilibrium at which P^{min} is the set of participants. We show that s^{min} is a strong equilibrium. By Lemma 3, it is sufficient to show that there is no deviation that satisfies (2).

Let T be a coalition and let s_T be a profile of strategies for T. Let $T_I^{min} = \{i \in T | s_i^{min} = I\}$ and $T_I = \{i \in T | s_i = I\}$. Note that the set of participants at (s_T, s_{-T}^{min}) is $(P^{min} \cup (T_I \backslash T_I^{min})) \backslash (T_I^{min} \backslash T_I)$. Let us denote this set as $\widetilde{P}$.

If T deviates in a way that satisfies $\theta_{\widetilde{P}} > c$, then we must have $\theta_{\widetilde{P}} \geq \theta_{P^{min}} > c$ because $\theta_{P^{min}}$ is the smallest sum of participants' benefits that is attained at strict Nash equilibria. Then, we have $\theta_{T_I^{min} \backslash T_I} \leq \theta_{T_I \backslash T_I^{min}}$. Hence, T cannot deviate in a way that satisfies (2). ■

From Lemma 3 and Proposition 2, a strong equilibrium exists, and the set of strong equilibria is contained in the set of strict Nash equilibria in the participation game. Therefore, the strong equilibrium concept is also a refinement that guarantees the existence of efficient Nash equilibria in this game.

Remark 2 From Proposition 1 and Lemma 3, the set of strict Nash equilibria and the set of efficient Nash equilibria coincide, and the set of strong equilibria is included in these equilibrium sets under the assumption that $y^P = 0$ for all P with $\theta_P = c$. This relationship between the three equilibrium sets still holds if agents can use randomized strategies. Proposition 1 holds because the set of strict Nash equilibria consists solely of pure-strategy profiles. Strong equilibria must be efficient Nash equilibria; hence, the strong equilibria must be strict Nash equilibria. We can also confirm that mixed-strategy Nash equilibria are inefficient since all of the efficient Nash equilibria are strict.

Remark 3 Let us consider the case in which, for all $P \subseteq N$, if $\theta_P = c$, $y^P = 1$. Then, there is not necessarily a strict Nash equilibrium in the participation game, as we confirmed in Remark 1. However, we can similarly show that the game has a strong equilibrium and the set of strong equilibria is included in the set of efficient Nash equilibria in this participation game.

3.3 Cost-sharing rules that guarantee the existence of efficient Nash equilibria

From Lemma 2, if the cost of the project is distributed according to the proportional cost-sharing rule, then the participation game has an efficient Nash equilibrium. Clearly, for every efficient cost-sharing rule, if P with $\theta_P > c$ is a Nash-equilibrium set of participants, then P satisfies $\theta_{P\setminus\{i\}} \leq c$ for all $i \in P$. However, it depends on the cost-sharing rule whether the condition $\theta_{P\setminus\{i\}} \leq c$ for all $i \in P$ is a sufficient condition for P to be attained at a Nash equilibrium. For example, when the equal cost-sharing rule is adopted, a P that satisfies $\theta_{P\setminus\{i\}} \leq c$ for all $i \in P$ is not necessarily supported as a Nash equilibrium. If the proportional cost-sharing rule is used, then this condition is essentially sufficient. Since there is a set of participants that satisfies the condition under Assumption 3, an efficient equilibrium exists if the proportional cost-sharing rule is used. We prove that the existence of efficient equilibria is guaranteed if the cost-sharing rule satisfies the following three conditions: for every $P \subseteq N$,

(Budget Balance) $\sum_{i\in P} x_i^P = c$ if $y^P = 1$.

(Individual Rationality) $V_i(y^P, x_i^P) \geq 0$ for every $i \in P$ if $y^P = 1$.

(Positive Cost Burden) $x_i^P > 0$ for every $i \in P$ if $y^P = 1$.

Note that the proportional cost-sharing rule satisfies these three conditions.

Proposition 3 Suppose that Assumption 2 holds. If the cost-sharing rule satisfies budget balance, individual rationality, and the positive cost burden condition, then the participation game has an efficient Nash equilibrium for every $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n_{++}$ such that $\theta_N > c$.

Proof. We show the existence of efficient Nash equilibria in both the case in which, for every $Q \subseteq N$, if $\theta_Q = c$, then $y^Q = 0$ and the case in which there is $Q \subseteq N$ such that $\theta_Q = c$ and $y^Q = 1$. Suppose that the cost-sharing rule satisfies budget balance, individual rationality, and the positive cost burden condition.

Let us consider the first case. Let P be a set of participants such that $\theta_P > c$. Obviously, if P is a Nash-equilibrium set of participants, then P satisfies $\theta_P - \theta_i \leq c$ for all $i \in P$. Now, we show that, if P satisfies this condition, then P is supported as a Nash equilibrium. From individual rationality, we have $\theta_i - x_i^P \geq 0$ for every $i \in P$. Therefore, every member of P does not have an incentive to deviate from I to O. From the positive cost burden condition, we have $\theta_j > \theta_j - x_j^{P \cup \{j\}}$ for every $j \notin P$. Hence, $j \notin P$ does not want to deviate from O to I. As a result, in the first case, for every P with $\theta_P > c$, P is a Nash-equilibrium set of participants if and only if P satisfies $\theta_P - \theta_i \leq c$ for all $i \in P$. Since such a set of participants P exists, the participation game has an efficient Nash equilibrium in the first case.

In the second case, we can similarly prove that Q with $\theta_Q = c$ and $y^Q = 1$ is supportable as a Nash equilibrium. Hence, an efficient Nash equilibrium also exists in the participation game in the second case. ■

Since the proportional cost-sharing rule is in the class of cost-sharing rules that satisfy the three conditions, we can say that the proportional cost-sharing rule is one of the favorable rules for the existence of efficient Nash equilibria.

Note that efficient Nash equilibria are not always strict Nash equilibria even if the cost-sharing rule satisfies the above three conditions. However, every efficient Nash equilibrium is a strict Nash equilibrium if $y^Q = 0$ for every Q with $\theta_Q = c$ and the condition of individual rationality is replaced with strict individual rationality, which requires that $\theta_i > x_i^P$ for every P with $\theta_P > c$ and every $i \in P$. It is noteworthy that the proportional cost-sharing rule satisfies strict individual rationality.

4 Participation games with a multi-unit public good

4.1 A participation game in which at most two units of the public good can be produced

In this section, we consider a participation game with a multi-unit public good. The participation game with a multi-unit public good consists of two stages. In the first stage, agents simultaneously choose I or O, and the agents that select I choose the level of the public good and share the cost of the public good in the second stage. However, in the participation game with a multi-unit public good, the level of the public good is assumed to be zero, one, or two. Let Y be a *public good space* such that $Y = \{(y_1, y_2) \in \{0,1\}^2 | y_1 \geq y_2\}$: if $y_1 = y_2 = 1$, then two units of the public good are produced; if $y_1 = 1$ and $y_2 = 0$, then one unit of the public good is produced; if $y_1 = y_2 = 0$, then zero units of the public good are produced. Let $c > 0$ be a constant cost of producing one unit of the public good. Let $\theta_i^k > 0$ denote agent i's marginal benefit from the k-th unit of the public good. Each agent i has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$. We denote $\theta_P^k = \sum_{j \in P} \theta_j^k$ for all $k \in \{1, 2\}$ and for all $P \subseteq N$. We assume that marginal benefits of all agents are decreasing.

Assumption 5 For all $i \in N$, $\theta_i^1 > \theta_i^2$.

Since the analyses are similar to those in Section 3 in the case in which one-unit public good provision is efficient, we focus on the case in which two units of the public good are produced at every Pareto efficient allocation.

Assumption 6 $\theta_N^2 > c$.

We set $\theta_i^0 > c$ for all $i \in N$ and $\theta_P^0 = \sum_{i \in P} \theta_i^0$ for all $P \subseteq N$ so that (C.1) of Assumption 7 below is well-defined. The following is the assumption regarding the second-stage outcomes.

Assumption 7 Let P be a set of participants. Let $(y^P, (x_j^P)_{j \in P})$ be the allocation for the participants. The allocation satisfies the following conditions.

(C.1) $y^P = \max\{k \in \{0, 1, 2\} \mid \theta_P^k - c > 0\}$. (Surplus Maximization)
(C.2) $\sum_{j \in P} x_j^P = y^P c$. (Budget Balance)
(C.3) $V_i(y^P, x_i^P) \geq 0$ for every $i \in P$. (Individual Rationality)
(C.4) $x_i^P > 0$ for every $i \in P$. (Positive Cost Burden)

From (C.1), the participants produce the public good in a way that maximizes the surplus of the participants. The cost of the public good is distributed in a way that satisfies the conditions of budget balance and individual rationality. Every participant pays a positive share of the cost.

Many allocation rules satisfy (C.1), (C.2), (C.3), and (C.4). For example, the unit-by-unit proportional cost-sharing rule introduced by Yu (2005) satisfies all these conditions. For every unit of the public good, the unit-by-unit proportional cost-sharing rule allocates the cost proportional to each agent's willingness to pay for that unit. Yu (2005) constructed a mechanism that implements the unit-by-unit cost-sharing rule.

In the participation game with a multi-unit public good, there is not necessarily a Nash equilibrium in which efficient allocations are attained. In the following example, there is no Nash equilibrium that supports an efficient allocation.

Example 2 Let $N = \{1, 2, 3, 4\}$. Suppose that $\theta_i^1 = 2$ and $\theta_i^2 = 0.8$ for all $i \in N$ and $c = 1$. The cost of the public good is assumed to be distributed equally among participants. Let P be a set of participants. Note that one unit of the public good is produced if $\#P = 1$, and two units of the public good are provided if $\#P \geq 2$. Table 2 shows the payoffs to participants and non-participants in this example. From the table, we can easily find that one and only one agent selects participation at every strict Nash equilibrium and that the allocation at the strict equilibrium is inefficient.

⟨Insert Table 2 here.⟩

4.2 Existence of Nash equilibria that support efficient allocations

In this subsection, we investigate whether a Nash equilibrium supports an efficient allocation in a participation game with a multi-unit participation game. We first characterize the set of Nash equilibria in which two units of the public good are produced.

Lemma 5 Suppose that Assumptions 2 and 7 are satisfied. The set of participants $P \subseteq N$ is supported as a Nash equilibrium and two units of the public good are provided at the equilibrium if and only if P satisfies (i) $\theta_P^2 > c$, (ii) $\theta_P^2 - \theta_i^2 \leq c$ for all $i \in P$, and (iii) if there is an agent $i \in P$ such $\theta_P^1 - \theta_i^1 > c$, then $\theta_i^2 \geq x_i^P$.

Proof. **(sufficiency)** Let P denote a set of participants that satisfies (i), (ii), and (iii). By (i), two units of the public good are produced if P is the set of participants. Let $i \in P$ be such that $\theta_P^1 - \theta_i^1 \leq c$. On the one hand, if i chooses I, then his payoff is $\sum_{k=1}^2 \theta_i^k - x_i^P$. On the other hand, if i chooses O, then his payoff is 0. It follows from (C.3) that i does not have an incentive to deviate from I to O. Let $j \in P$ be such that $\theta_P^1 - \theta_j^1 > c$. Participant j receives payoff $\sum_{k=1}^2 \theta_j^k - x_j^P$ if he chooses I, and he obtains payoff θ_j^1 otherwise. Since $\theta_j^2 \geq x_j^P$ from (iii), j does not have an incentive to switch from I to O. No non-participant $h \notin P$ has an incentive to choose I because $x_h^{P \cup \{h\}} > 0$ from (C.4) and two units of the public good are provided irrespective of h's participation decision. Hence, P is a set of participants that is supportable as a Nash equilibrium.

(necessity) Suppose that the set of participants P is supported as a Nash equilibrium and two units of the public good are provided at the equilibrium. Since two units of the public good are provided, P must satisfy condition (i). The set P must obviously satisfy (ii) because P is supported as an equilibrium. Let $j \in P$ be such that $\theta_P^1 - \theta_j^1 > c$. We must have $\theta_j^2 \geq x_j^P$ because j obtains payoff θ_j^1 if he chooses O, and he receives payoff $\sum_{k=1}^2 \theta_j^k - x_j^P$ if he chooses I. Thus, P satisfies (iii). ■

We now determine whether two units of the public good are produced at a Nash equilibrium. First, consider the following case:

Case 1 There exists a set of participants P such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$.

Lemma 6 In Case 1, there is a set of participants that is supported as a Nash equilibrium of the participation game when Assumptions 2, 5, and 7 hold.

Proof. Let $P \subseteq N$ be such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. Note that $\theta_P^2 - \theta_i^2 \leq \theta_P^1 - \theta_i^1 \leq c$ for every $i \in P$ with the first inequality holding with equality if $P \setminus \{i\}$ is empty. Hence, P is a Nash-equilibrium set of participants, and two units of the public good are provided. ■

Next, we consider Case 2:

Case 2 For every $P \subseteq N$, if P satisfies $\theta_P^2 > c$, then $\theta_{P\setminus\{i\}}^1 > c$ for some $i \in P$.

Lemma 7 Suppose that Assumptions 2, 5, and 7 hold. Let $P \subseteq N$ be a set of participants such that $\theta_{P\setminus\{i\}}^1 > c$ for *every* $i \in P$ and $\theta_P^2 > c$. Then, P is a Nash-equilibrium set of participants if and only if $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for every $i,\ j \in P$.

Proof. Let $P \subseteq N$ be a set of participants such that $\theta_{P\setminus\{i\}}^1 > c$ for every $i \in P$ and $\theta_P^2 > c$. Since $\theta_{P\setminus\{i\}}^1 > c$ for every $i \in P$, we have $\#P \geq 2$.

(sufficiency) Let us suppose that P satisfies $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for every $i,\ j \in P$. Then, we have $V_i(y^P, x_i^P) = \theta_i^1 + \theta_i^2 - x_i^P = \theta_i^1$ and $V_i(y^{P\setminus\{i\}}, x_i^{P\setminus\{i\}}) = \theta_i^1$ for every $i \in P$. From these conditions, P is supported as a Nash equilibrium of the participation game.

(necessity) Suppose that P is supported as a Nash equilibrium of the participation game. Suppose, without loss of generality, that $P = \{1, 2, \ldots, l\}$ with $l \geq 2$ and $\theta_1^2 \geq \theta_2^2 \geq \cdots \geq \theta_l^2$. Then, there is $\alpha_i \in (0, 1]$ for every $i \in P$ such that $\theta_i^2 = \alpha_i \theta_1^2$. Note that $1 = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l > 0$.

From Lemma 5, P satisfies the following conditions:

$$\sum_{i\in P}\alpha_i\theta_1^2 > c. \tag{6}$$

$$\left(\sum_{i\in P}\alpha_i - \alpha_j\right)\theta_1^2 \le c \text{ for every } j\in P. \tag{7}$$

$$\alpha_j\theta_1^2 \ge x_j^P \text{ for every } j\in P. \tag{8}$$

We obtain from (6) that $\theta_1^2 > c/\sum_{i\in P}\alpha_i$. Since $\sum_{i\in P}\alpha_i - \alpha_l \ge \sum_{i\in P}\alpha_i - \alpha_j$ for every $j\in P$, (7) implies $\theta_1^2 \le c/(\sum_{i\in P}\alpha_i - \alpha_l)$. It follows from (8) that $\theta_1^2 \ge x_j^P/\alpha_j$ for every $j \in P$. By these conditions, we must have

$$\frac{c}{\sum_{i\in P\setminus\{l\}}\alpha_i} - \frac{x_j^P}{\alpha_j} \ge 0 \text{ for every } j\in P \tag{9}$$

so that P satisfies (6), (7), and (8). It follows from (9) that

$$\frac{1}{\alpha_j\sum_{i\in P\setminus\{l\}}\alpha_i}\left(\alpha_j c - x_j^P\sum_{i\in P\setminus\{l\}}\alpha_i\right) \ge 0 \text{ for every } j\in P.$$

We obtain from these conditions that $\alpha_j c - x_j^P\sum_{i\in P\setminus\{l\}}\alpha_i \ge 0$ for every $j\in P$. Summing these conditions for every $j\in P$ yields

$$\sum_{j\in P}\alpha_j c \ge \left(\sum_{j\in P}x_j^P\right)\left(\sum_{i\in P\setminus\{l\}}\alpha_i\right) = 2c\sum_{i\in P\setminus\{l\}}\alpha_i.$$

Therefore, we have

$$\alpha_l \ge \sum_{i\in P\setminus\{l\}}\alpha_i. \tag{10}$$

First, we prove that $\#P = 2$ and $\theta_i^2 = \theta_j^2$ for all $i,\ j\in P$. Suppose, on the contrary, that $\#P\ge 3$. Since $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_l$, we have $\alpha_l < \sum_{j\in P\setminus\{l\}}\alpha_j$, a contradiction. Therefore, it follows that $\#P = 2$, which implies that $l = 2$ and $\alpha_2\ge\alpha_1$. The condition that $\alpha_2\ge\alpha_1$ together with the inequality $\alpha_1\ge\alpha_2$ implies that $\alpha_1 = \alpha_2$. Therefore, we have $\theta_1^2 = \theta_2^2$.

Second, we show that $\theta_1^2 = \theta_2^2 = c$ and $x_1^P = x_2^P = c$. Let $\theta^2 = \theta_1^2 = \theta_2^2 = c$. Since P is a Nash-equilibrium set of participants, θ^2 satisfies $2\theta^2 > c$, $\theta^2 \leq c$, $\theta^2 \geq x_1^P$, and $\theta^2 \geq x_2^P$. From the first two conditions, we have $\theta^2 \in (c/2, c]$. Thus, $x_1^P \leq c$ and $x_2^P \leq c$ because $\theta^2 \leq c$. Since $x_1^P + x_2^P = 2c$, we have $x_1^P = x_2^P = c$. Hence, $\theta^2 = c$, and so P is supported as a Nash equilibrium of the participation game. ■

Corollary 1 Suppose that Assumptions 2, 5, and 7 hold. Suppose that the agents' preferences are identical: for every $k \in \{1,2\}$ and for every $i,\ j \in N$, $\theta_i^k = \theta_j^k$. Then, in Case 2, there is a Nash equilibrium that supports an efficient allocation if and only if there is a set of participants P such that $\#P = 2$, $\theta_i^2 = \theta_j^2 = c$, and $x_i^P = x_j^P = c$ for all $i,\ j \in P$.

Proof. Suppose that the agents' preferences are identical. Then, for every $P \subseteq N$ with $\theta_P^2 > c$, if $\theta_P^1 - \theta_i^1 > c$ for some $i \in P$, then $\theta_P^1 - \theta_i^1 > c$ for every $i \in P$. Thus, Corollary 1 follows immediately from Lemma 7. ■

From Lemma 6 and Corollary 1, we can demonstrate that, in the case of identical agents, the public good is less likely to be provided efficiently if the participation of many agents is needed for efficient provision of the public good. Suppose that all agents receive a (marginal) benefit $\theta^k > 0$ from the k-th unit of the public good for every $k \in \{1,2\}$. Let $(\theta^1, \theta^2) \in \mathbb{R}_{++}^2$ be a profile of the marginal benefits. Note that $\theta^1 > \theta^2$.

Proposition 4 Suppose that Assumptions 2, 5, and 7 are satisfied. For each $p \geq 1$, the following are necessary and sufficient conditions under which there are Nash equilibria that support the participation of p agents and the provision of two units of the public good.

(1) $\theta^2 > c$ when $p = 1$.

(2) either (2.1) or (2.2) holds when $p = 2$

  (2.1) $c \geq \theta^1 > \theta^2 > \dfrac{c}{2}$ and $1 > \dfrac{\theta^2}{\theta^1} > \dfrac{1}{2}$.

  (2.2) $\theta^1 > \theta^2 = c$ and each of the two participants pays c.

(3) $\dfrac{c}{p-1} \geq \theta^1 > \theta^2 > \dfrac{c}{p}$ and $1 > \dfrac{\theta^2}{\theta^1} > 1 - \dfrac{1}{p}$ when $p \geq 3$.

Proof. Note that $\frac{c}{p-1} \geq \theta^1 > \theta^2 > \frac{c}{p}$ implies $1 > \frac{\theta^2}{\theta^1} > \frac{c}{p} \Big/ \frac{c}{p-1} = 1 - \frac{1}{p}$ for $p \geq 2$. It is immediate from Lemma 6 that (1) is the necessary and sufficient condition if $p = 1$. Next, consider the case of $p = 2$. Clearly, there is a Nash equilibrium which supports two-agent participation and two-unit provision of the public good if either (2.1) or (2.2) holds. Conversely, if there is such a Nash equilibrium, then $2\theta^2 > c$. If $\theta^1 \leq c$, we have $c \geq \theta^1 > \theta^2 > c/2$. If $\theta^1 > c$, then we obtain that $\theta^1 > \theta^2 = c$, and each of the two participants pays c from Corollary 1, which is (2.2). Hence, either (2.1) or (2.2) is satisfied. Finally, let us consider the case of $p \geq 3$. From Corollary 1, if p agents choose I and two units of the public good are provided at a Nash equilibrium, then we have $(p-1)\theta^1 \leq c$. We also obtain $p\theta^2 > c$, since two-unit provision of the public good is attained at a Nash equilibrium. Conditions $(p-1)\theta^1 \leq c$ and $p\theta^2 > c$, together with $\theta^1 > \theta^2$, imply that $\frac{c}{p-1} \geq \theta^1 > \theta^2 > \frac{c}{p}$. The converse direction is obvious. ■

From Proposition 4, we can show the set of parameters at which the public good is provided efficiently in Figure 1. In (1) of Proposition 4, one agent chooses participation, and two units of the public good are provided in a Nash equilibrium if and only if $\theta^2 > c$. The set of profiles (θ^1, θ^2) that satisfies $\theta^2 > c$ is shown in (a) of Figure 1. In (2) of Proposition 4, efficient provision of the public good is achieved if either one of the following conditions is satisfied:

(i) $\theta^1 \leq c$ and $\theta^2 > \frac{c}{2}$
(ii) $\theta^1 > \theta^2 = c$

The set of (θ^1, θ^2) that satisfies (i) or (ii) is depicted in Figure 1 (b): (θ^1, θ^2) in the shaded triangle satisfies (i) and B in Figure 1 (b) is the set of preference parameters that satisfy (ii). In (3) of Proposition 4, efficient provision of the public good is supportable as a Nash equilibrium if and only if (θ^1, θ^2) satisfies $\theta^1 \leq \frac{c}{p-1}$ and $\theta^2 > \frac{c}{p}$. The shaded area in Figure 1 (c) is the range of (θ^1, θ^2) for which the public good is provided efficiently.

Note that, under the condition of $(p-1)\theta^1 > c$, efficient provision of the public good is attained at an equilibrium only if $p = 2$. From Corollary 1, a Nash equilibrium supports efficient provision of the public good only if $\theta^2 = c$, even though $p = 2$ and $(p-1)\theta^1 > c$ are satisfied. These observations indicate that it is almost impossible for a Nash equilibrium to support two-unit provision of the public good if $(p-1)\theta^1 > c$. Thus, the range of (θ^1, θ^2) for which a Nash equilibrium achieves two-unit provision of the public good consists largely of (θ^1, θ^2), which satisfies $p\theta^2 > c \geq (p-1)\theta^1$ and is depicted as a triangular area in Figure 1 (b) and (c). The triangular areas satisfy $1 > \dfrac{\theta^2}{\theta^1} > 1 - \dfrac{1}{p}$ for $p \geq 2$.

Since $1 - \dfrac{1}{p}$ converges to 1 as the number of participants p becomes large, the range of the ratio of marginal benefits $\dfrac{\theta^2}{\theta^1}$ shrinks as the number of participants increases. Thus, when the set of participants consists of many agents, the ratio of the marginal benefits must be high for the set to be supported as a Nash equilibrium. We can say from this result that efficient provision of two units of the public good is less likely to occur as a Nash-equilibrium outcome if the number of agents is large and efficiency requires a large fraction of agents to participate.

⟨ Insert Figure 1 here. ⟩

The difficulty with two-unit provision of the public good stems from the following facts: first, in the case in which $(p-1)\theta^1 \leq c < p\theta^2$, the participation game has an efficient Nash equilibrium. However, the set of (θ^1, θ^2) that satisfies $(p-1)\theta^1 \leq c$ and $p\theta^2 > c$ shrinks as p gets large. Second, in the case in which $(p-1)\theta^1 > c$ and $p\theta^2 > c \geq (p-1)\theta^2$, if $\theta^2 \geq x_i^P$ for every $i \in P$, there is an efficient Nash equilibrium in this game. However, the inequality $\theta^2 \geq x_i^P$ for every $i \in P$ is hard to satisfy. Summing up $\theta^2 \geq x_i^P$ for all $i \in P$ yields $p\theta^2 \geq 2c$, which means that the sum of marginal benefits for the participants from a second unit of the public good covers the cost of two units of the public good. The conditions $(p-1)\theta^1 > c$, $p\theta^2 > c \geq (p-1)\theta^2$, and $p\theta^2 \geq 2c$ are compatible only when $p = 2$.

Our result has an implication for the difficulty of providing the public good efficiently in a participation game when more than two units of the public good can be provided. If more than two units of the public good can be provided and the efficient amount of the public good is more than two units, then a Nash-equilibrium set of participants that produces the public good efficiently must satisfy more conditions than a Nash-equilibrium set of participants with two-unit efficient provision. For example, consider the case in which the provision of three units of the public good is efficient. Let θ_i^3 be a benefit which i receives from the third unit of the public good, and let $\theta_P^3 := \sum_{j \in P} \theta_j^3$. If P is a set of participants that produces three units of the public good and is supported as a Nash equilibrium, then P satisfies (a) $\theta_P^3 > c$, (b) $\theta_P^3 - \theta_i^3 \leq c$ for every $i \in P$, (c-1) if there is $i \in P$ such that $\theta_P^2 - \theta_i^2 > c$, then $\theta_i^3 \geq x_i^P$, and (c-2) if there is $i \in P$ such that $\theta_P^1 - \theta_i^1 > c \geq \theta_P^2 - \theta_i^2$, then $\theta_i^2 + \theta_i^3 \geq x_i^P$. Condition (c-1) would be severer than (iii) of Proposition 4 under the assumption of diminishing marginal benefits. In addition, the set of participants must satisfy (c-2). Therefore, we can say that the conditions for efficient provision of the public good at Nash equilibria when the efficient amount of the public good is three units are more stringent than those when the efficient level is two units. In general, if $y^* \geq 3$ designates the efficient amount of the public good, then a Nash-equilibrium set of participants that produces y^* units of the public good satisfies: for every $y \in \{0, 1, \ldots, y^* - 1\}$, if there is $i \in P$ such that $\theta_P^y - \theta_i^y > c \geq \theta_P^{y+1} - \theta_i^{y+1}$, then $\sum_{k=y+1}^{y^*} \theta_i^k \geq x_i^P$. From this condition, the conditions for efficient provision at equilibrium become severer as y^* becomes larger. Thus, we can infer that efficient provision of more than two units is less likely to occur than that of two units in equilibrium.

Remark 4 In Proposition 6, we characterized a Nash-equilibrium set of participants P under the condition that $\theta_P^1 - \theta_i^1 > c$ for every $i \in P$. This condition holds only if agents' preferences are identical or slightly different. However, a set of participants does not necessarily satisfy this condition if the set of participants is composed of agents who have different preferences. Thus, in the case of heterogeneous agents, there may be a set of participants P that satisfies $\theta_P^2 > c$, $\theta_{P \setminus \{i\}}^2 \leq c$ for every $i \in P$, $\theta_{P \setminus \{i\}}^1 > c$ for some $i \in P$, and $\theta_{P \setminus \{j\}}^1 < c$ for some $j \in P$. The following examples indicate that such sets of participants may or may not be Nash-equilibrium sets of participants in the participation game, depending on the preference parameters of the participants.

Example 3 Let $N = \{1, 2\}$, and let $\theta_1^1 = 40$, $\theta_1^2 = 9$, $\theta_2^1 = 7$, $\theta_2^2 = 6$, and $c = 10$.

In this example, the costs of producing the public good are distributed according to a unit-by-unit proportional cost-sharing rule among participants: for every unit of the public good, the unit-by-unit proportional cost-sharing rule allocates the cost proportional to each agent's willingness to pay for that unit. Two units of the public good are produced only if two agents choose I, and one unit of the public good is provided only when agent 1 chooses I and agent 2 chooses O. If agent 1 and agent 2 choose I, then the payoff of agent 1 is $\theta_1^1 + \theta_1^2 - \frac{\theta_1^1}{\theta_1^1 + \theta_2^1} c - \frac{\theta_1^2}{\theta_1^2 + \theta_2^2} c = \frac{1621}{47} \approx 34.49$, and that of agent 2 is $\theta_2^1 + \theta_2^2 - \frac{\theta_2^1}{\theta_1^1 + \theta_2^1} c - \frac{\theta_2^2}{\theta_1^2 + \theta_2^2} c = \frac{353}{47} \approx 7.51$. Table 3 is the payoff matrix of this example. In this example, there is a Nash equilibrium at which two agents choose I and two units of the public good are provided.

⟨ Insert Table 3 here.⟩

Example 4 Consider a two-agent participation game in which $\theta_1^1 = 12$, $\theta_1^2 = 8$, $\theta_2^1 = 8$, $\theta_2^2 = 6$, and $c = 10$. As in Example 3, the costs of producing the public good are allocated according to the unit-by-unit proportional cost-sharing rule among participants. The payoff matrix is shown in Table 4. In this game, there is only one Nash equilibrium in which agent 1 chooses I and agent 2 chooses O. At this equilibrium, one unit of the public good is produced and an inefficient allocation is obtained.

⟨ Insert Table 4 here.⟩

Agents have substantially different preferences for the public good in Example 3, while they have relatively similar preferences in Example 4. From these examples, it seems valid to conjecture that efficiency is attained at a Nash equilibrium if the agents' preferences differ greatly.

5 Conclusion

We have investigated a participation game for the provision of a discrete public good and provided conditions for its efficient provision. First, we examined the case of a public project. In this case, we first provided refinements of Nash equilibria that support efficient allocations. We showed that the set of strict Nash equilibria coincides with the set of efficient Nash equilibria and the set of strong equilibria is included in the set of strict Nash equilibria. Thus, strict Nash equilibria and strong equilibria ensure efficient equilibria. This result is useful for discussing whether one type of equilibrium is more focal than another. In the participation game with a public project, we also provided a sufficient condition for cost-sharing rules under which efficient Nash equilibria exist. As a consequence, we found that the proportional cost-sharing rule is one of the favorable rules for the existence of efficient Nash equilibria. Second, we examined a case in which at most two units of the public good are provided. In this case, there is not necessarily a Nash equilibrium that supports an efficient allocation. We proved that, in the case of identical agents, the set of participants consisting of many agents is not likely to be a Nash-equilibrium set of participants. Therefore, efficient equilibria are unlikely to exist if the number of agents in the economy is large and efficiency requires a large fraction of agents to participate. This result implies the difficulty to provide a discrete and multi-unit public good efficiently in general environments.

In the case of heterogeneous agents, it is unclear which sets of participants are supported as Nash equilibria and which conditions assure efficient provision of the public good at Nash equilibria. Future studies will be needed to establish conditions under which efficient allocations are attained at Nash equilibria.

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	I	O
I	5/12, 5/12, 5/12	1/4, 3/4, 1/4
O	3/4, 1/4, 1/4	0, 0, 0

I

	I	O
I	1/4, 1/4, 3/4	0, 0, 0
O	0, 0, 0	0, 0, 0

O

Table. 1 Payoff matrix of Example 1

The number of participants	Payoffs to participants	Payoffs to non-participants
0	-	0
1	1	2
2	1.8	2.8
3	32/15	2.8
4	2.3	-

Table. 2 Payoffs of Example 2

2 \ 1	I	O
I	34.49, 7.51	30, 7
O	0, 0	0, 0

Table. 3 The payoff matrix of Example 3

1 \ 2	I	O
I	8.29, 5.71	2, 8
O	0, 0	0, 0

Table. 4 The payoff matrix of Example 4

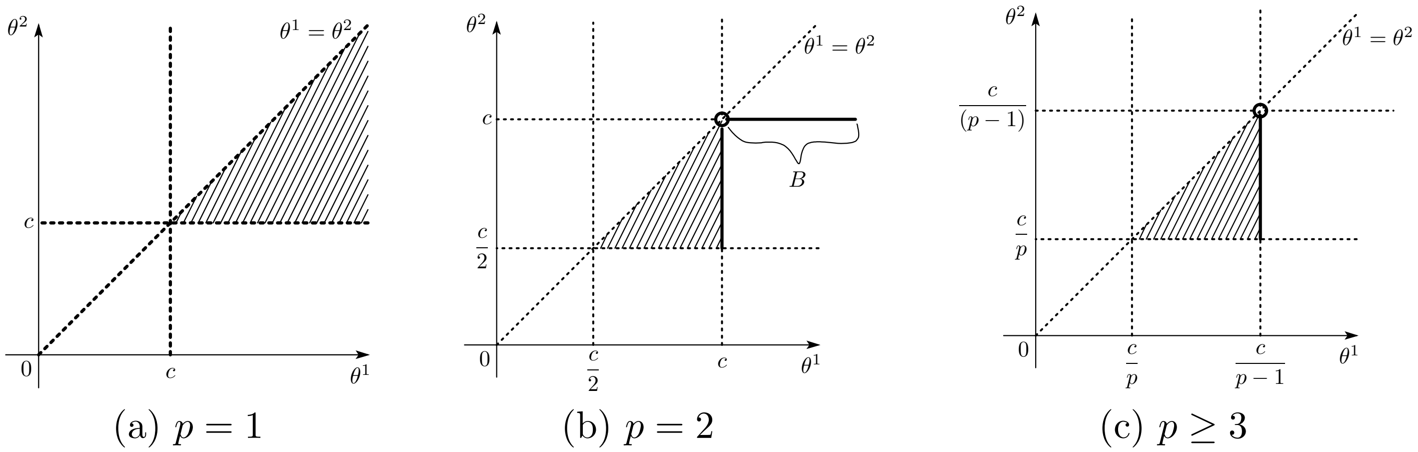

Fig. 1 Efficient provision of the public good